On the geometry of biharmonic maps and biharmonic submanifolds

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Chen definition

Let

\[ i : M \hookrightarrow \mathbb{R}^n \]

be the canonical inclusion and \( \mathbf{H} = (H_1, \ldots, H_n) \) the mean curvature vector field.

**Definition (B-Y. Chen)** A submanifold \( M \subset \mathbb{R}^n \) is **biharmonic** iff

\[ \Delta \mathbf{H} = (\Delta H_1, \ldots, \Delta H_n) = 0 \]

where \( \Delta \) is the Beltrami-Laplace operator on \( M \) w.r.t. the metric induced by \( i \).
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- Why biharmonic?

\[ m \Delta H = \Delta(-\Delta i) = -\Delta^2 i \]
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where $\Delta$ is the Beltrami-Laplace operator on $M$ w.r.t. the metric induced by $i$.

- **Why biharmonic?**

  $$m \Delta H = \Delta(-\Delta i) = -\Delta^2 i$$

- **CMC submanifolds, $|H| = \text{constant}$, are not necessarily biharmonic.**
Biharmonic submanifolds in $\mathbb{E}^n(c)$

Let

\[ i : M^m \to \mathbb{E}^n(c) \]

be the canonical inclusion of a submanifold $M$ in a constant sectional curvature $c$ manifold.
Biharmonic submanifolds in $\mathbb{E}^n(c)$

Let

$$i : M^m \rightarrow \mathbb{E}^n(c)$$

be the canonical inclusion of a submanifold $M$ in a constant sectional curvature $c$ manifold.

**Definition** $M$ is a biharmonic submanifold iff

$$\Delta^i H = m c H$$

where

- $H \in C(i^{-1}(T\mathbb{E}^n(c)))$ denotes the mean curvature vector field of $M$ in $\mathbb{E}^n(c)$

- $\Delta^i$ is the rough Laplacian on $i^{-1}(T\mathbb{E}^n(c))$
Remark

If $E^n(c) = S^n$ then one can consider $S^n \subset \mathbb{R}^{n+1}$ and the inclusion

$$i : M^m \rightarrow S^n \subset \mathbb{R}^{n+1}$$

can be seen as a map into $\mathbb{R}^{n+1}$. 
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**Alternative problem** (Alias, Barros, Ferrández)

$$\Delta H' = (\Delta H_1, \ldots, \Delta H_{n+1}) = \lambda H'$$

where $H'$ is the mean curvature vector field of the inclusion as a map into $\mathbb{R}^{n+1}$. 
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This is NOT the biharmonic condition for submanifolds in \( S^n \)

\[
\Delta H' = m \cdot c \cdot H' \iff \Delta^i H = m \cdot c \cdot H
\]
Where does this definition come from?

To understand the origin of the biharmonic equation we need to use the theory of harmonic maps.
The energy Functional

Harmonic maps $\varphi : (M, g) \to (N, h)$ are critical points of the energy

$$E(\varphi) = \frac{1}{2} \int_M |d\varphi|^2 v_g$$

and they are solutions of the Euler-Lagrange equation

$$\tau(\varphi) = \text{trace}_g \nabla d\varphi = 0$$
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- If \( \varphi \) is an isometric immersion, with mean curvature vector field \( \mathbf{H} \), then:

\[
\tau(\varphi) = m\mathbf{H}
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The bienergy Functional

The bienergy functional (proposed by Eells–Lemaire) is

\[ E_2(\varphi) = \frac{1}{2} \int_M |\tau(\varphi)|^2 v_g \]

Critical points of \( E_2 \) are called biharmonic maps and they are solutions of the Euler-Lagrange equation (Jiang):

\[ \tau_2(\varphi) = -\Delta^\varphi \tau(\varphi) - \text{trace}_g R^N (d\varphi, \tau(\varphi))d\varphi = 0 \]

where \( R^N \) is the curvature operator on \( N \).
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- The biharmonic equation

\[ \tau_2(\varphi) = 0 \]

is a fourth-order non-linear elliptic equation (not easy to solve!).
Non existence of Biharmonic Maps

Harmonic maps are trivially biharmonic
A non harmonic biharmonic map is called proper biharmonic
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$$\int_M |\nabla \tau|^2 v_g = \int_M \langle \Delta^\varphi \tau, \tau \rangle v_g = - \text{trace} \int_M \langle R^N (d\varphi, \tau) d\varphi, \tau \rangle v_g \leq 0$$
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\( \Rightarrow \nabla \tau = 0. \)
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$\Rightarrow \nabla \tau = 0$. From

$$
div \langle \tau, d\varphi \rangle = \text{trace} \langle \nabla \tau, d\varphi \rangle + |\tau|^2
$$
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$\Rightarrow \nabla \tau = 0$. From

$$0 = \int_M \text{div} \langle \tau, d\varphi \rangle v_g = \int_M |\tau|^2 v_g$$

$\Rightarrow \tau = 0$. Q.E.D.
Remarks

- $M$ compact $+$ $\text{Sec}^N \leq 0 \Rightarrow$ there exists a harmonic map

$$\varphi : M \rightarrow N$$

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- Harmonic maps do not always exists. There exists no harmonic map from

  \[ \mathbb{T}^2 \to S^2 \]

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**Problem** Find biharmonic maps $\mathbb{T}^2 \to S^2$ of degree $\pm 1$

- So far we only know examples of biharmonic maps $\mathbb{T}^2 \to S^2$ whose image is a curve.
Examples of proper biharmonic maps
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- Any polynomial map of degree 3 between Euclidean spaces
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**Property (Almansi):** let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) be any harmonic function then

\[
g(x) = |x|^\ell f(x)
\]

is proper biharmonic for any harmonic \( f \) if and only if \( \ell = 2 \).
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- From the Hopf map \( H : \mathbb{C}^2 \to \mathbb{R} \times \mathbb{C} \) we get the proper biharmonic map

\[
\mathbb{C}^2 \to \mathbb{R} \times \mathbb{C}, \ (z, w) \mapsto (|z|^2 + |w|^2)(|z|^2 - |w|^2, 2z\bar{w})
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- Let \( f(x_1, ..., x_n) = \sum_{i=1}^{n} a_i x_i, \ a_i \in \mathbb{R}, \) then

\[ g(x) = |x|^{2-n} f(x) \]

is proper biharmonic \( \) (M–Impera)
Examples of proper biharmonic maps

- The generalized Kelvin transformation

$$\varphi : \mathbb{R}^m \setminus \{0\} \to \mathbb{R}^m \setminus \{0\}, \quad \varphi(p) = \frac{p}{|p|^{\ell}}$$

is proper biharmonic iff $\ell = m - 2$ \hspace{1cm} (B–M–O)
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- The **quaternionic** multiplication

  \[ \mathbb{H} \rightarrow \mathbb{H}, \quad q \mapsto q^n \]

  is biharmonic for any \( n \in \mathbb{N} \) \hspace{3em} (Fueter, 1935)
Lets go back to biharmonic submanifolds

If $\varphi : M \rightarrow \mathbb{E}^n(c)$ is an isometric immersion then

$$\tau(\varphi) = mH, \quad \tau_2(\varphi) = -m\Delta^\varphi H + c m^2 H$$

thus $\varphi$ is biharmonic iff

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Let's go back to biharmonic submanifolds

If $\varphi : M \rightarrow \mathbb{E}^n(c)$ is an isometric immersion then

$$\tau(\varphi) = m H, \quad \tau_2(\varphi) = -m \Delta \varphi H + cm^2 H$$

thus $\varphi$ is biharmonic iff

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Moreover, if $\varphi : M \rightarrow \mathbb{R}^n$ is anisometric immersion, set $\varphi = (\varphi_1, \ldots, \varphi_n)$ and $H = (H_1, \ldots, H_n)$. Then

$$\Delta \varphi H = (\Delta H_1, \ldots, \Delta H_n)$$

and we recover Chen's definition.
Biharmonic submanifolds of $\mathbb{E}^n(c)$
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**Proposition** [Chen ($c = 0$) - Oniciuc ($c \leq 0$)] Let

$$\varphi : M \to \mathbb{E}^n(c)$$

be an isometric immersion with $|H| = \text{constant}$. If $c \leq 0$, then $\varphi$ is biharmonic iff $H = 0$. 
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**Proof**

Biharmonic $\Rightarrow \Delta \varphi H = m \, c \, H$. 
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**Proposition** [Chen ($c = 0$) - Oniciuc ($c \leq 0$)] Let \( \varphi : M \to \mathbb{E}^n(c) \) be an isometric immersion with $|H| = \text{constant}$. If $c \leq 0$, then $\varphi$ is biharmonic iff $H = 0$.

**Proof** Biharmonic $\Rightarrow \Delta \varphi H = mcH$. Replacing in the Weitzenböck formula
\[
\frac{1}{2} \Delta \varphi |H|^2 = \langle \Delta \varphi H, H \rangle - |\nabla H|^2
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we get, since $c \leq 0$,
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|\nabla H|^2 = mc|H|^2 \leq 0
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Thus we conclude that $\nabla H = 0$. 
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Thus we conclude that $\nabla H = 0$.
Next, for an isometric immersion we have:
\[
|H|^2 = -\text{trace}(d\varphi, \nabla H)
\]
Q.E.D.
Geometric conditions for biharmonic submanifolds

By decomposing the equation \( \Delta \varphi H = mcH \) in its normal and tangent components we find that an isometric immersion \( \varphi : M^m \to \mathbb{E}^n (c) \) is biharmonic iff

\[
\begin{cases}
-\Delta^\perp H - \text{trace } B(\cdot, A_H \cdot) + mcH = 0 \quad \text{(normal)} \\
2 \text{trace } A_{\nabla^\perp (\cdot)} H(\cdot) + \frac{m}{2} \text{grad}(|H|^2) = 0 \quad \text{(tangent)}
\end{cases}
\]

\( A \) is the Weingarten operator - \( B \) the second fundamental form \( \nabla^\perp \) and \( \Delta^\perp \) the connection and the Laplacian in the normal bundle.
Geometric conditions for biharmonic submanifolds

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For hypersurfaces

$$\begin{cases} 
\Delta^\perp H - (mc - |A|^2)H = 0 \\
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Non existence of biharmonic submanifolds
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**Proposition** [Chen \((c = 0)\), Caddeo–M–O \((c \leq 0)\)] If \(c \leq 0\), there exists no proper biharmonic surfaces \(M^2 \subset \mathbb{E}^3(c)\).
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Proof
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**Proposition** [Chen ($c = 0$), Caddeo–M–O ($c \leq 0$)] If $c \leq 0$, there exists no proper biharmonic surfaces $M^2 \subset \mathbb{E}^3(c)$.

**Proof** In this case

$$H = |H|\eta,$$

where $\eta$ is a unit normal to $M$. 
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Proposition [Chen \((c = 0)\), Caddeo–M–O \((c \leq 0)\)] If \(c \leq 0\), there exists no proper biharmonic surfaces \(M^2 \subset \mathbb{E}^3(c)\).

Proof In this case

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\mathbf{H} = |H|\eta,
\]

where \(\eta\) is a unit normal to \(M\).

Then, biharmonicity, Gauss and Codazzi equations imply that \(|H|\) is a solution of a polynomial equation with constant coefficients, thus \(|\mathbf{H}|\) is constant.

Q.E.D.
Chen’s Conjecture

**Conjecture**

Biharmonic submanifolds of $\mathbb{E}^n(c)$, $n > 3$, $c \leq 0$, are minimal
Chen’s Conjecture

**Conjecture**

_Biharmonic submanifolds of \( \mathbb{E}^n(c), \ n > 3, \ c \leq 0, \) are minimal_

Partial solutions of the conjecture are known for:

- curves of \( \mathbb{R}^n \) (Dimitric)
- submanifolds of finite type in \( \mathbb{R}^n \) (Dimitric)
- hypersurfaces with at most two principal curvatures (B–M–O)
- pseudo-umbilical submanifolds \( M^m \subset \mathbb{E}^n(c), \ c \leq 0, \ m \neq 4, \) (Caddeo–M–O, Dimitric)
- hypersurfaces of \( \mathbb{E}^4(c), \ c \leq 0 \) (Hasanis–Vlachos, B–M–O)
- spherical submanifolds of \( \mathbb{R}^n \) (Chen)
- submanifolds of bounded geometry (Ichiyama–Inoguchi–Urakawa)
All the non existence results described in the previous section do not hold for submanifolds in the sphere.

**Problem:**

*Classify all biharmonic submanifolds of $S^n$*
Main examples of biharmonic submanifolds in $\mathbb{S}^n$
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B1. The small hypersphere

$S^m \left( \frac{1}{\sqrt{2}} \right) \xrightarrow{\text{biharmonic}} S^{m+1}$
Main examples of biharmonic submanifolds in $S^n$

**B1.** The small hypersphere

\[ S^m \left( \frac{1}{\sqrt{2}} \right) \overset{\text{biharmonic}}{\longrightarrow} S^{m+1} \]

**B2.** The standard products of spheres

\[ S^{m_1} \left( \frac{1}{\sqrt{2}} \right) \times S^{m_2} \left( \frac{1}{\sqrt{2}} \right) \overset{\text{biharmonic}}{\longrightarrow} S^{m+1} \]

\[ m_1 + m_2 = m - 1 \text{ and } m_1 \neq m_2. \]
Main examples of biharmonic submanifolds in $S^n$

B3. Composition property

$M^m \overset{\text{minimal}}{\longrightarrow} S^{n-1}(\frac{1}{\sqrt{2}}) \overset{\text{biharmonic}}{\longrightarrow} S^n$

proper biharmonic
Main examples of biharmonic submanifolds in $S^n$

**B3. Composition property**

\[ M^m \rightarrow S^{n-1} \left( \sqrt{\frac{1}{2}} \right) \rightarrow S^n \]

minimal \quad biharmonic

\[ \text{proper biharmonic} \]

**B4. Product composition property**

\[ M_1^{m_1} \times M_2^{m_2} \rightarrow S^{n_1} \left( \sqrt{\frac{1}{2}} \right) \times S^{n_2} \left( \sqrt{\frac{1}{2}} \right) \rightarrow S^n \]

minimal

\[ \text{proper biharmonic} \]

\[ n_1 + n_2 = n - 1, \quad m_1 \neq m_2 \]
Biharmonic curves in $S^n$ (Caddeo–M–O)
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Biharmonic curves in $S^n$ (Caddeo–M–O)

- $\gamma \subset S^n$ biharmonic curve
- $\kappa = \text{curvature}$
- $\kappa = 1$
- $S^1(\frac{1}{\sqrt{2}}) \to S^2$
Biharmonic curves in $S^n$ (Caddeo–M–O)

\begin{align*}
\gamma \subset S^n \text{ biharmonic curve} \\
\kappa = \text{curvature} \\
\kappa = 1 \\
\kappa \in (0,1) \\
S^1\left(\frac{1}{\sqrt{2}}\right) \longrightarrow S^2
\end{align*}
Biharmonic curves in $S^n$ (Caddeo–M–O)

$\gamma \subset S^n$ biharmonic curve

$\kappa = \text{curvature}$

$\kappa = 1$

$S^1(\frac{1}{\sqrt{2}}) \rightarrow S^2$

$\kappa \in (0, 1)$

geo

$\gamma \rightarrow S^1(\frac{1}{\sqrt{2}}) \times S^1(\frac{1}{\sqrt{2}}) \rightarrow S^3$

biharmonic

$\gamma$ geodesic of slope $\neq \pm 1, 0 \text{ and } \infty$
Biharmonic hypersurfaces in $S^n$
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Biharmonic hypersurfaces in $S^n$

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B1
Biharmonic hypersurfaces in $\mathbb{S}^n$

$M^m \subset \mathbb{S}^{m+1}$ biharmonic

$\kappa = \text{number of distinct principal curvature}$

$\kappa = 1$

$\mathbb{S}^m \left(\frac{1}{\sqrt{2}}\right) \rightarrow \mathbb{S}^{m+1}$

$\kappa \leq 2$

B1
Biharmonic hypersurfaces in $\mathbb{S}^n$

$M^m \subset \mathbb{S}^{m+1}$ biharmonic

$\kappa = \text{number of distinct principal curvature}$

$\kappa = 1$
$\kappa \leq 2$

$\mathbb{S}^m \left( \frac{1}{\sqrt{2}} \right) \rightarrow \mathbb{S}^{m+1}$ B1

$\mathbb{S}^{m_1} \left( \frac{1}{\sqrt{2}} \right) \times \mathbb{S}^{m_2} \left( \frac{1}{\sqrt{2}} \right) \rightarrow \mathbb{S}^{m+1}$ B2
Biharmonic hypersurfaces in $S^n$

$M^m \subset S^{m+1}$ biharmonic

$k = \text{number of distinct principal curvature}$

$k = 1$

$k \leq 2$

$k = 3$

$S^m (\frac{1}{\sqrt{2}}) \rightarrow S^{m+1}$

$B1$

$S^{m_1} (\frac{1}{\sqrt{2}}) \times S^{m_2} (\frac{1}{\sqrt{2}}) \rightarrow S^{m+1}$

$B2$
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- $\kappa = 1$
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- $\kappa \leq 2$
- $\kappa = 3$ - Compact + CMC
  - $S^{m_1}(\frac{1}{\sqrt{2}}) \times S^{m_2}(\frac{1}{\sqrt{2}}) \rightarrow S^{m+1}$ \textbf{B2}
Biharmonic hypersurfaces in $S^n$

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Non Existence
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$\mathbb{S}^m (\frac{1}{\sqrt{2}}) \rightarrow \mathbb{S}^{m+1}$

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B2

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Non-Existence

Isoparametric
Biharmonic hypersurfaces in $\mathbb{S}^n$

$M^m \subset \mathbb{S}^{m+1}$ biharmonic

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$\mathbb{S}^{m\left(\frac{1}{\sqrt{2}}\right)} \rightarrow \mathbb{S}^{m+1}$

B1

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B2

$\kappa = 3$

Compact + CMC

Non Existence

Isoparametric

Ichiyama-Inoguchi-Urakawa
CMC Biharmonic submanifolds in $S^n$
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$M^3 \subset S^4 \rightarrow M^m \subset S^n$ biharmonic + CMC
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Compact

$|H| = 1$

$|H| \in (0, 1)$
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$M^3 \subset S^4 \quad \Rightarrow \quad M^m \subset S^n$ biharmonic + CMC

Compact

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$M^m$ minimal $\Rightarrow S^{n-1}(\frac{1}{\sqrt{2}})$ biharmonic $\Rightarrow S^n$

B3
CMC Biharmonic submanifolds in $S^n$

$M^3 \subset S^4$ \quad $M^m \subset S^n$ biharmonic + CMC

Compact \quad |H| = 1 \quad |H| \in (0, 1)

$M^m$ minimal \quad $S^{n-1}(\frac{1}{\sqrt{2}})$ biharmonic \quad $S^n$

B3

Compact
CMC Biharmonic submanifolds in $S^n$

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Compact \quad $|H| = 1$ \quad $|H| \in (0, 1)$

$M^m$ minimal \quad $S^{n-1}(\frac{1}{\sqrt{2}})$ biharmonic

$S^n$ B3 \quad $M$ is of 2-type
CMC Biharmonic submanifolds in $\mathbb{S}^n$

$M^3 \subset \mathbb{S}^4 \quad \rightarrow \quad M^m \subset \mathbb{S}^n$ biharmonic + CMC

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$M^m$ minimal $\mathbb{S}^{n-1}(\frac{1}{\sqrt{2}})$ biharmonic $\mathbb{S}^n$

B3

$M$ is of 2-type

$M$ is of 1-type
CMC Biharmonic submanifolds in $\mathbb{S}^n$

$M^3 \subset \mathbb{S}^4 \rightarrow M^m \subset \mathbb{S}^n$ biharmonic + CMC

Compact

$|H| = 1$  $|H| \in (0, 1)$

$M^m \subset \mathbb{S}^{n-1}(\frac{1}{\sqrt{2}})$ biharmonic  $\rightarrow \mathbb{S}^n$

B3

Compact

$M$ is of 2-type

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$M$ is of 2-type
Biharmonic submanifolds in $S^n$ with $\nabla^\perp H = 0$

$M^m \subset S^n$ biharmonic + PMC
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Biharmonic submanifolds in $S^n$ with $\nabla^\perp H = 0$

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B3
Biharmonic submanifolds in $S^n$ with $\nabla^\perp H = 0$

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$\nabla^\perp B = 0$

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B3
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$\nabla^\perp B = 0$

$M^{m_1}_{1} \times M^{m_2}_{2} \overset{\text{min}}{\longrightarrow} S^{n_1}(\frac{1}{\sqrt{2}}) \times S^{n_2}(\frac{1}{\sqrt{2}}) \longrightarrow S^n$

$B3$

$B4$
Pseudo-umbilical biharmonic submanifolds in $S^n$

CMC proper biharmonic submanifolds with $|H| = 1$ in $S^n$ are B3 and they are pseudo-umbilical:

$$A_H = |H|^2 \text{Id}$$
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**Question** When a proper biharmonic pseudo-umbilical submanifold in $S^n$ has $|\mathbf{H}| = 1$, thus B3?
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**Question** When a proper biharmonic pseudo-umbilical submanifold in $S^n$ has $|H| = 1$, thus B3?

**Theorem** Let $M^m$ be a compact pseudo-umbilical submanifold in $S^n$, $m \neq 4$. Then $M$ is proper biharmonic if and only if $M$ is B3.
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The examples of Sasahara et al

**Theorem** Let $\varphi : M^3 \to S^5$ be a proper biharmonic anti-invariant immersion. Then the position vector field $x_0 = x_0(u, v, w)$ in $\mathbb{R}^6$ is given by

$$x_0(u, v, w) = e^{iw}(e^{iu}, ie^{-iu} \sin \sqrt{2}v, ie^{-iu} \cos \sqrt{2}v)$$

Moreover, $|H| = 1/3$. 
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**Theorem** Let $\phi : M^2 \to S^5$ be a proper biharmonic Legendre immersion. Then the position vector field $x_0 = x_0(u, v)$ of $M$ in $\mathbb{R}^6$ is given by:

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\]

The immersion \( \phi \) is NOT PMC
Conjecture
The only proper biharmonic hypersurfaces in $S^n$ are B1 or B2.

Conjecture
Any biharmonic submanifold in $S^n$ has constant mean curvature.
Remark

An isometric immersion $\varphi : M^m \to \mathbb{E}^n(c)$ is biharmonic iff

$$\begin{align*}
-\Delta^\perp H - \text{trace } B(\cdot, A_H \cdot) + mcH &= 0 \quad \text{normal} \\
2 \text{trace } A_{\nabla^\perp(\cdot)} H(\cdot) + \frac{m}{2} \text{grad}(|H|^2) &= 0 \quad \text{tangent}
\end{align*}$$

Most of the classification results described depend only on the tangent part of $\tau_2$. 
Remark

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$$
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Has the condition

$$\tau_2(\varphi)^\top = 0$$

a variational meaning?
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Has the condition

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YES
The stress-energy tensor

As described by Hilbert, the stress-energy tensor associated to a variational problem is a symmetric 2-covariant tensor field $S$ conservative at critical points, i.e. $\text{div } S = 0$ at these points.
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  \[ S = \frac{1}{2} |d\varphi|^2 g - \varphi^* h, \quad \text{div } S = -\langle \tau(\varphi), d\varphi \rangle \]
  (Baird–Eells)

- For biharmonic maps the stress-energy tensor is
  \[ S_2(X, Y) = \frac{1}{2} |\tau(\varphi)|^2 \langle X, Y \rangle + \langle d\varphi, \nabla \tau(\varphi) \rangle \langle X, Y \rangle \]
  \[ -\langle d\varphi(X), \nabla_Y \tau(\varphi) \rangle - \langle d\varphi(Y), \nabla_X \tau(\varphi) \rangle \]
  with
  \[ \text{div } S_2 = -\langle \tau_2(\varphi), d\varphi \rangle \]
  (Jiang, Loubeau–M–O)
The meaning of $S_2 = 0$ (Loubeau–M–O)
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A smooth map $\varphi : (M, g) \rightarrow (N, h)$ is biharmonic if it is a critical points of the bienergy w.r.t. variations of the map.
The meaning of \( S_2 = 0 \) (Loubeau–M–O)

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\[
F : G \rightarrow \mathbb{R}, \quad F(g) = E_2(\varphi),
\]

where \( G \) is the set of Riemannian metrics on \( M \)
The meaning of $S_2 = 0$ (Loubeau–M–O)

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**Theorem**

$$\delta(F(g_t)) = -\frac{1}{2} \int_M \langle S_2, \omega \rangle \, v_g,$$

The tensor $S_2$ vanishes precisely at critical points of the energy (bienergy) for variations of the domain metric, rather than for variations of the map.
The condition $S_2 = 0$ is rather strong, in fact

$$S_2 = 0 \implies \text{harmonic}$$

if:

- $\dim(M) = 2$
- $M$ is compact and orientable with $\dim(M) \neq 4$
- $\varphi$ is an isometric immersion and $\dim(M) \neq 4$
- $M$ is complete and $\varphi$ has finite energy and bienergy
If $\varphi : (M, g) \rightarrow (N, h)$ is an isometric immersion from

$$\text{div } S_2 = -\langle \tau_2(\varphi), d\varphi \rangle$$

\[\Downarrow\]

$$\text{div } S_2 = -\tau_2(\varphi)^T$$
If $\varphi : (M, g) \rightarrow (N, h)$ is an isometric immersion from

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$\downarrow$

$$\text{div} S_2 = -\tau_2(\varphi)^\top$$

---

**Problem**

Study isometric immersions in space forms with $\text{div} S_2 = 0$
Biharmonic submanifolds in a Riemannian manifold

An isometric immersion

\[ \varphi : (M, g) \rightarrow (N, h) \]

is biharmonic iff

\[
\begin{align*}
\Delta^\perp H + \text{trace } B(\cdot, A_H \cdot) + \text{trace}(R^N(\cdot, H)\cdot)^\perp &= 0 \\
\frac{m}{2} \text{grad } |H|^2 + 2 \text{trace } A_{\nabla^\perp(\cdot)} H(\cdot) + 2 \text{trace}(R^N(\cdot, H)\cdot)^\top &= 0
\end{align*}
\]
Results for Bih. Sub. in non constant sec. curv. manifolds

- In three-dimensional homogeneous spaces (Thurston’s geometries)
  (Inoguchi, Ou–Wang, Caddeo–Piu–M–O)

- There exists examples of proper biharmonic hypersurfaces in a space with negative non constant sectional curvature
  (Ou–Tang)

- It is initiated the study of biharmonic submanifolds in complex space forms

- There are several works on biharmonic submanifolds in contact manifold and Sasakian space forms
  (Inoguchi, Fetcu–O, Sasahara)
In a Sasakian manifold 

\[(N, \Phi, \xi, \eta, g)\]

a submanifold \(M \subset N\) tangent to \(\xi\) is called anti-invariant if \(\Phi\) maps any tangent vector to \(M\), which is normal to \(\xi\), to a vector which is normal to \(M\).
Finite $k$-type submanifolds

An isometric immersion $\phi : M \to \mathbb{R}^{n+1}$ ($M$ compact) is called of finite $k$-type if

$$\phi = \phi_0 + \phi_1 + \cdots + \phi_k$$

where

$$\Delta \phi_i = \lambda_i \phi_i, \quad i = 1, \ldots, k$$

and $\phi_0 \in \mathbb{R}^{n+1}$ is the center of mass

A submanifold $M \subset S^n \subset \mathbb{R}^{n+1}$ is said to be of finite type if it is of finite type as a submanifold of $\mathbb{R}^{n+1}$.

A non null finite type submanifold in $S^n$ is said to be mass-symmetric if the constant vector $\phi_0$ of its spectral decomposition is the center of the hypersphere $S^n$, i.e. $\phi_0 = 0$. 