Biharmonic maps between Riemannian manifolds

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The definition

Let \( \varphi: (M, g) \to (N, h) \) be a smooth map between Riemannian manifolds.

Biharmonic maps are critical points of the energy functional

\[
E_{\varphi} = \frac{1}{2} \int_M |d\varphi|^2 \, g
\]

The corresponding Euler-Lagrange equation is

\[
\tilde{\nabla}^2 \varphi = 0
\]

where \( \tilde{\nabla}^2 \) is the rough Laplacian on \( M \) and \( \varphi \) is a smooth map between Riemannian manifolds.

The classification results

Proper biharmonic curves on a surface of revolution are classified, for example we have (2):

Proper biharmonic curves on the eight-dimensional Thurston geometries are classified by means of Cartan-Feinstein metrics

\[
\varphi: (M, g) \to (N, h)
\]

and their parametrization is described explicitly in (6).

In particular:

- A proper biharmonic surface of \( S^3 \) is locally a piece of \( S^3 \). If compact, it is \( S^3 \).
- A Hopf cylinder \( S^3 \) is a proper biharmonic surface of \( S^3 \).

The stability

Let \( \varphi: (M, g) \to (N, h) \) be a biharmonic map. Then the Hessian of the bienergy at \( \varphi \) is given by

\[
\nabla^2 \varphi = \frac{1}{2} \int_M |d\varphi|^2 \, g
\]

and we deduce

a) If \( m = 1 \) and \( n \neq 0 \), then nullity(\( d\varphi \)) = 0.

b) If \( m \neq 0 \), then nullity(\( d\varphi \)) = 8.

The general properties

A harmonic map is obviously a biharmonic map and an absolute minimum of the bienergy. A non-harmonic biharmonic map is called proper biharmonic.

A proper biharmonic map does not exist:

- If \( M \) is compact and \( \text{Riem}^N \leq 0 \) [10].
- If \( M \to N \) is an isometric immersion with \( \text{rank} \, R^N \leq 0 \) and \( \text{Riem}^N \leq 0 \) and \( N \) has constant non-positive curvature [9, 12].
- If \( \varphi \) is a Riemannian submersion with basic tension field and one of the following holds [14]:
  - \( M \) is compact, orientable and \( \text{Riem}^N \leq 0 \)
  - \( \text{Riem}^N \leq 0 \) and \( \varphi \) is not proper
  - \( N \) is compact and \( \text{Riem}^N < 0 \)

General Chen’s Conjecture: (Biharmonic submanifolds of a manifold \( N \) with \( \text{Riem}^N \leq 0 \) are minimal)

The energy tensor

In the context of harmonic maps, the stress-energy tensor is

\[
S(\varphi, d\varphi) = \frac{1}{2} \sqrt{\det(g)} \nabla \varphi \otimes d\varphi
\]

For biharmonic maps the stress-energy tensor is

\[
S_{\varphi}(X, Y) = \frac{1}{2} \frac{\det(\varphi)}{\det(g)} \left( \frac{d\varphi(X)}{d\varphi(Y)} \right)_X + \frac{1}{2} \sqrt{\det(g)} \nabla \varphi \otimes d\varphi
\]

Property: The composition \( \varphi: M \to N \) is biharmonic if and only if and it is pseudo-umbilical.

Application: There exist closed orientable embedded proper biharmonic surfaces of arbitrary genus in \( S^4 \).

Classification

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Further studies

The hypersurfaces \( S^3 \) and \( S^3 \) and the generalized Clifford torus are the only known examples of proper biharmonic hypersurfaces.

Open problem: Prove that proper biharmonic hypersurfaces of \( S^3 \) have constant mean curvature.

Open problem: Compute the biharmonic index of the generalized Clifford torus in \( S^3 \).

Harmonic maps do not always exist, for instance J. Eells and J.C. Wood showed that there exists no harmonic map from \( S^2 \) to \( S^4 \) (whatever the metrics chosen) in the homotopy class of the Hopf degree 2.

Open problem: Find a biharmonic map from \( S^2 \) to \( S^4 \).

The bibliography of biharmonic maps

http://birecmedia.unica.it/biharmap/