THE CLASSIFICATION OF BIHARMONIC CURVES OF CARTAN-VRANCEANU 3-DIMENSIONAL SPACES

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ABSTRACT. In this article we characterize all biharmonic curves of the Cartan-Vranceanu 3-dimensional spaces and we give their explicit parametrizations.

1. INTRODUCTION

Biharmonic curves $\gamma : I \subset \mathbb{R} \to (N, h)$ of a Riemannian manifold are the solutions of the fourth order differential equation

$$\nabla^2 \gamma' - R(\gamma', \nabla \gamma') \gamma' = 0.$$  

As we shall detail in the next section, they arise from a variational problem and are a natural generalization of geodesics.

In the last decade have appeared several papers on the construction and classification of biharmonic curves starting with [4], where the authors described the case of curves of a surface. Biharmonic curves in a 3-dimensional Riemannian manifold with constant sectional curvature $K \leq 0$ are geodesics (see [8], for $K = 0$, and in [3], for $K < 0$); while, in [2], the authors proved that, for $K > 0$, the biharmonic curves are helices, that is curves with constant geodesic curvature and geodesic torsion.

Among the 3-dimensional manifolds of non-constant sectional curvature a special role is played by the homogeneous Riemannian spaces with a large isometry group. For these spaces, except for those with constant negative curvature, there is a nice local representation given by the following two-parameter family of Riemannian metrics (the Cartan-Vranceanu metric)

$$ds^2_{\ell,m} = \frac{dx^2 + dy^2}{1 + m(x^2 + y^2)} + \left( dz + \frac{\ell}{2} \frac{ydx - xdy}{1 + m(x^2 + y^2)} \right)^2,$$

defined on $M = \mathbb{R}^3$ if $m \geq 0$, and on $M = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 < -\frac{1}{m}\}$ otherwise.

Biharmonic curves on $(M, ds^2_{\ell,m})$ have been already studied for particular values of $\ell$ and $m$. In particular, if $m = 0$ and $\ell \neq 0$, $(M, ds^2_{\ell,m})$ is the Heisenberg space $\mathbb{H}_3$ endowed with a left invariant metric and the explicit solutions of the biharmonic curves were obtained in [5]; if $m = 1$ and $\ell \neq 0$ a study of the biharmonic curves was given also in [7].

In this paper we prove that the biharmonic curves of $(M, ds^2_{\ell,m})$ are helices and we find out their explicit parametric equations, for all values of $\ell$ and $m$.  

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2. Preliminary

2.1. Biharmonic curves. Harmonic maps $\phi : (M, g) \to (N, h)$ between Riemannian manifolds are the critical points of the energy functional $E(\phi) = \frac{1}{2} \int_M |d\phi|^2 v_g$, and the corresponding Euler-Lagrange equation is given by the vanishing of the tension field $\tau(\phi) = \text{trace} \nabla d\phi$. Biharmonic maps (as suggested by J. Eells and J.H. Sampson in [9]) are the critical points of the bienergy functional $E_2(\phi) = \frac{1}{4} \int_M |\tau(\phi)|^2 v_g$. In [11] G.Y. Jiang derived the first variation formula of the bienergy showing that the Euler-Lagrange equation for $E_2$ is

$$\tau_2(\phi) = -J(\tau(\phi)) - \Delta \tau(\phi) - \text{trace} R^N(d\phi, \tau(\phi)) d\phi = 0,$$

where $J$ is the Jacobi operator of $\phi$ and $R^N(X, Y) = \nabla_X \nabla Y - \nabla_Y \nabla X - \nabla [X, Y]$. The equation $\tau_2(\phi) = 0$ is called the biharmonic equation.

Since $J$ is linear, any harmonic map is biharmonic. Therefore, the main interest is to find and classify proper biharmonic maps, that is non-harmonic biharmonic maps.

In this paper we restrict our attention to curves $\gamma : I \to (N, h)$ parametrized by arc length, from an open interval $I \subset \mathbb{R}$ to a Riemannian manifold. In this case, putting $T = \gamma'$, the tension field becomes $\tau(\gamma) = \nabla_T T$ and the biharmonic equation reduces to

$$\nabla^3_T T - R(T, \nabla_T T)T = 0.$$ (1)

To describe geometrically Equation (1) let recall the definition of the Frenet frame.

**Definition 2.1** (See, for example, [14]). The Frenet frame $\{E_i\}_{i=1,\ldots,n}$ associated to a curve $\gamma : I \subset \mathbb{R} \to (N^n, h)$, parametrized by arc length, is the orthonormalisation of the $(n+1)$-uple $\{\nabla^{(k)}_\gamma d\gamma(\frac{\partial}{\partial t})\}_{k=0,\ldots,n}$, described by:

- $F_1 = d\gamma(\frac{\partial}{\partial t}),$
- $\nabla^\gamma F_1 = k_1 F_2,$
- $\nabla^\gamma F_i = -k_{i-1} F_{i-1} + k_i F_{i+1}, \quad \forall i = 2, \ldots, n-1,$
- $\nabla^\gamma F_n = -k_{n-1} F_{n-1},$

where the functions $\{k_1, k_2, \ldots, k_{n-1}\}$ are called the curvatures of $\gamma$ and $\nabla^\gamma$ is the connection on the pull-back bundle $\gamma^{-1}(TN)$. Note that $F_1 = T = \gamma'$ is the unit tangent vector field along the curve.

Using the Frenet frame, the biharmonic equation (1) reduces to a differential system of the curvatures of $\gamma$ as shown in the following

**Proposition 2.2.** Let $\gamma : I \subset \mathbb{R} \to (N^n, h)$ $(n \geq 2)$ be a curve parametrized by arc length from an open interval of $\mathbb{R}$ into a Riemannian manifold $(N, g)$. Then $\gamma$ is biharmonic if and only if:

$$\begin{cases}
k_1 k'_1 = 0 \\
k''_1 - k_1^3 - k_1 k'_2 + k_1 R(F_1, F_2, F_1, F_2) = 0 \\
2k'_2 + k_1 k'_2 + k_1 R(F_1, F_2, F_1, F_3) = 0 \\
k_1 k_2 k_3 + k_1 R(F_1, F_2, F_1, F_4) = 0 \\
k_1 R(F_1, F_2, F_1, F_j) = 0 & j = 5, \ldots, n
\end{cases}$$
BIHARMONIC CURVES

Proof. With respect to its Frenet frame, the biharmonic equation of \( \gamma \) is:

\[
\nabla^3 F_1 - R(F_1, \nabla F_1)F_1 = -3k_1k_1'F_1 + (k_1'' - k_1^3 - k_1k_2^2)F_2 + (2k_1'k_2 + k_1k_2')F_3 + k_1k_2k_3F_4 - k_1R(F_1, F_2)F_1 = 0.
\]

\( \square \)

If we look for proper biharmonic solutions, that is for biharmonic curves with \( k_1 \neq 0 \), we have

\[
\begin{cases}
  k_1 = \text{constant} \neq 0 \\
  k_1^2 + k_2^2 = R(F_1, F_2, F_1, F_2) \\
  k_2 = -R(F_1, F_2, F_3) \\
  k_2k_3 = -R(F_1, F_2, F_1, F_4) \\
  R(F_1, F_2, F_1, F_j) = 0 & j = 5, \ldots, n
\end{cases}
\]

2.2. Riemannian structure of Cartan-Vranceanu 3-manifolds. Let \( m \) be a real parameter. We shall denote by \( M \) the whole \( \mathbb{R}^3 \) if \( m \geq 0 \), and by \( M = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 < -\frac{1}{m}\} \) otherwise. Consider on \( M \) the following two-parameter family of Riemannian metrics

\[
(3) \quad ds_{\ell,m}^2 = \frac{dx^2 + dy^2}{[1 + m(x^2 + y^2)]^2} + \left( dz + \frac{\ell ydx - xdy}{2[1 + m(x^2 + y^2)]}\right)^2,
\]

where \( \ell, m \in \mathbb{R} \).

These metrics have been known for a long time. They can be found in the classification of 3-dimensional homogeneous metrics given by L. Bianchi in 1897 (see [1]); later, they appeared in form (3) in É. Cartan, ([6] p. 304) and in G. Vranceanu (see [15], p. 354). Their geometric interest lies in the following fact: the family of metrics (3) includes all 3-dimensional homogeneous metrics whose group of isometries has dimension 4 or 6, except for those of constant negative sectional curvature.

The Cartan-Vranceanu metric (3) can be written as:

\[
(4) \quad ds_{\ell,m}^2 = \sum_{i=1}^{3} \omega^i \otimes \omega^i
\]

where, putting \( F = 1 + m(x^2 + y^2) \),

\[
(4) \quad \omega^1 = \frac{dx}{F}; \quad \omega^2 = \frac{dy}{F}; \quad \omega^3 = dz + \frac{\ell ydx - xdy}{2F},
\]

and the orthonormal basis of dual vector fields to the 1-forms (4) is

\[
(5) \quad E_1 = F \frac{\partial}{\partial x} - \frac{\ell y}{2} \frac{\partial}{\partial z}; \quad E_2 = F \frac{\partial}{\partial y} + \frac{\ell x}{2} \frac{\partial}{\partial z}; \quad E_3 = \frac{\partial}{\partial z}.
\]

For completeness we give the expressions, with respect to the orthonormal basis (5), of the Levi-Civita connection, of the nonzero components of the curvature tensor and of the
Ricci tensor:
\[
\begin{align*}
\nabla E_1 E_1 &= 2myE_2 \\
\nabla E_2 E_2 &= 2mxE_1 \\
\nabla E_1 E_3 &= 0
\end{align*}
\]
(6)
\[
\begin{align*}
\nabla E_1 E_2 &= -2myE_1 + \frac{\ell}{2}E_3 \\
\nabla E_1 E_3 &= \nabla E_3 E_1 = -\frac{\ell}{2}E_2 \\
\nabla E_2 E_1 &= -2mxE_2 - \frac{\ell}{2}E_3 \\
\nabla E_2 E_3 &= \nabla E_3 E_2 = \frac{\ell}{2}E_1 \\
\nabla E_3 E_1 &= \nabla E_1 E_3 = \frac{\ell}{2}E_2 \\
\nabla E_3 E_2 &= \nabla E_2 E_3 = \frac{\ell}{2}E_1 
\end{align*}
\]
(7)
\[
\begin{align*}
R_{1212} &= 4m - \frac{3}{4}\ell^2 \\
R_{1313} &= \frac{\ell^2}{4} \\
R_{2323} &= \frac{\ell^2}{4}
\end{align*}
\]
(8)
\[
\rho_{11} = \rho_{22} = 4m - \frac{\ell^2}{2} \\
\rho_{33} = \frac{\ell^2}{2}.
\]

Remark 2.3.

- If \( \ell = 0 \), then \( M \) is the product of a surface \( S \) with constant Gaussian curvature \( 4m \) and the real line \( \mathbb{R} \).
- If \( \rho_{11} - \rho_{33} = 4m - \ell^2 = 0 \), then \( M \) has non negative constant sectional curvature.
- From the Kowalski’s classification [12] we know that the principal Ricci curvatures of \( SU(2) \) satisfy \( \rho_{33} > 0, \rho_{11} + \rho_{33} > 0 \) and \( \rho_{11} \neq \rho_{33} \). Thus, form (8), if \( \ell \neq 0 \) and \( m > 0 \), \( M \) is locally \( SU(2) \).
- Similarly, if \( \ell \neq 0 \) and \( m < 0 \), \( M \) is locally \( \tilde{SL}(2, \mathbb{R}) \), while if \( m = 0 \) and \( \ell \neq 0 \) we get the left invariant metric on the Heisenberg space \( \mathbb{H}_3 \).

Remark 2.3 gives rise to a nice geometric description of the metric \( ds^2_{\ell,m} \) as shown in Figure 1.

**Figure 1.** The geometric description of the metric \( ds^2_{\ell,m} \).

3. Biharmonicity conditions for curves in \((M, ds^2_{\ell,m})\)

Let \( \gamma : I \to (M, ds^2_{\ell,m}) \) be a differentiable curve parametrized by arc length and let \( \{F_1 = T = T_i E_i, F_2 = N = N_i E_i, F_3 = B = B_i E_i\} \) be the Frenet frame field tangent to \( M \) along \( \gamma \) decomposed with respect to the orthonormal basis (5).
By making use of (2) and of the expression of the curvature tensor field (7), we obtain the following system for the proper biharmonic curves

\[
\begin{align*}
  &k = \text{constant} \neq 0 \\
  &k^2 + \tau^2 = \frac{\ell^2}{4} - (\ell^2 - 4m)B_3^2 \\
  &\tau' = (\ell^2 - 4m)N_3B_3,
\end{align*}
\]

where \( k = k_1 \) and \( \tau = -k_2 \).

By analogy with curves in \( \mathbb{R}^3 \), also following [10], we keep the name helix for a curve in a Riemannian manifold having constant both geodesic curvature and geodesic torsion.

**Remark 3.1.**

(i) If \( \ell = m = 0 \), \( (M, ds^2_{\ell,m}) \) is the Euclidean space and \( \gamma \) is biharmonic if and only if it is a line (see [8]);

(ii) if \( \ell^2 = 4m \) and \( \ell \neq 0 \), then \( (M, ds^2_{\ell,m}) \) is locally the 3-dimensional sphere and the proper biharmonic curves were classified in [2], where it was proved that they are helices;

(iii) if \( \ell = 0 \) and \( m < 0 \), \( (M, ds^2_{\ell,m}) \) is isometric to \( \mathbb{H}^2 \times \mathbb{R} \) with the product metric and it can be show that all biharmonic curves are geodesics.

(iv) if \( m = 0 \) and \( \ell \neq 0 \), \( (M, ds^2_{\ell,m}) \) is the Heisenberg space \( \mathbb{H}_3 \) endowed with a left invariant metric and the biharmonic curves were studied in [5].

(v) if \( \ell = 1 \) a study of the explicit solutions of (9) was given in [7].

From now on we shall assume that \( \ell^2 \neq 4m \) and \( m \neq 0 \). This is, essentially, the only case left to study according to Remark 3.1.

As in previous cases we have

**Theorem 3.2.** If \( \gamma : I \to (M, ds^2_{\ell,m}) \) is a proper biharmonic curve parametrized by arc length, then it is a helix.

**Proof.** Let \( \gamma : I \to (M, ds^2_{\ell,m}) \) be a non geodesic curve parametrized by arc length. Then from the Frenet’s equation we have

\[ \langle \nabla_T B, E_3 \rangle = \tau N_3, \]

while, using the definition of covariant derivative, we get

\[ \langle \nabla_T B, E_3 \rangle = B_3' + \frac{\ell}{2}(T_1B_2 - T_2B_1) = B_3' - \frac{\ell}{2}N_3. \]

Comparing the two equations we have

\[ \tau N_3 = B_3' - \frac{\ell}{2}N_3. \]

Assume now that \( \gamma \) is a proper biharmonic curve. We first show that \( B_3 \neq 0 \). Indeed, if \( B_3 = 0 \), then (10) implies that

\[ N_3 (\tau + \frac{\ell}{2}) = 0. \]

The latter equation gives two possibilities:

- if \( N_3 = 0 \) (and \( B_3 = 0 \)) then \( T = \pm E_3 \) and the curve \( \gamma \) is a geodesic;
- if \( (\tau + \frac{\ell}{2}) = 0 \) then, using the second equation of (9), we must have that \( \gamma \) is again a geodesic.
Therefore, $B_3 \neq 0$. Deriving the second equation of (9) yields
\[ \tau' = - (\ell^2 - 4m) B_3 B_3' \]
and, taking into account the third equation of (9), gives
\[ (\ell^2 - 4m) B_3 (\tau N_3 + B_3') = 0. \]
Thus $\tau N_3 = -B_3'$ that summed with $\tau N_3 = B_3' - \frac{\ell}{2} N_3$ leads to
\[ N_3 (4\tau + \ell) = 0, \]
and consequently $\tau$ is constant.

From the proof of Theorem 3.2 and (9) we have, in conclusion,

**Corollary 3.3.** Let $\gamma : I \rightarrow (M, ds_{\ell,m}^2)$ be a curve parametrized by arc length. Then $\gamma$ is a proper biharmonic curve if and only if
\[
\begin{aligned}
& k = \text{constant} \neq 0 \\
& \tau = \text{constant} \\
& N_3 = 0 \\
& k^2 + \tau^2 = \frac{\ell^2}{4} - (\ell^2 - 4m) B_3^2.
\end{aligned}
\]

4. Explicit Formulas for Proper Biharmonic Curves in $(M, ds_{\ell,m}^2)$

In this section we use Corollary 3.3 to derive the explicit parametric equations of proper biharmonic curves in $(M, ds_{\ell,m}^2)$. We first prove the following

**Lemma 4.1.** Let $\gamma : I \rightarrow (M, ds_{\ell,m}^2)$ be a non-geodesic curve parametrized by arc length. If $N_3 = 0$, then
\[ T(t) = \sin \alpha_0 \cos \beta(t) E_1 + \sin \alpha_0 \sin \beta(t) E_2 + \cos \alpha_0 E_3, \]
where $\alpha_0 \in (0, \pi)$.

**Proof.** If $\gamma' = T = T_1 E_1 + T_2 E_2 + T_3 E_3$, from
\[
\nabla_T T = (T_1' + \ell T_2 T_3 + 2 m x T_3^2 - 2 m y T_1 T_2) E_1 \\
+ (T_2' - \ell T_1 T_3 + 2 m y T_1^2 - 2 m x T_1 T_2) E_2 + T_3' E_3 \\
= kN
\]
it follows that $N_3 = 0$ if and only if $T_3' = 0$, i.e. if and only if $T_3 = \text{constant}$. Since $T_3 \in (-1, 1)$ and the norm of $T$ is one, there exists a constant $\alpha_0 \in (0, \pi)$ and a unique (up to an additive constant $2k\pi$) smooth function $\beta$ such that
\[ T(t) = \sin \alpha_0 \cos \beta(t) E_1 + \sin \alpha_0 \sin \beta(t) E_2 + \cos \alpha_0 E_3 \]

Starting from the expression (12) of $T$ we are ready to state the main result.
Theorem 4.2. Let \((M, ds^2_{\ell,m})\) be the Cartan-Vranceanu space with \(m \neq 0\) and \(\ell^2 - 4m \neq 0\). Assume that \(\delta = \ell^2 + (16m - 5\ell^2)\sin^2 \alpha_0 \geq 0\), \(\alpha_0 \in (0, \pi)\), and denote by \(2\omega_{1,2} = -\ell \cos \alpha_0 \pm \sqrt{\delta}\). Then, the parametric equations of all proper biharmonic curves of \((M, ds^2_{\ell,m})\) are of the following three types.

**Type I**

\[
\begin{align*}
  x(t) &= b \sin \alpha_0 \sin \beta(t) + c, \quad b, c \in \mathbb{R}, \quad b > 0 \\
  y(t) &= -b \sin \alpha_0 \cos \beta(t) + d, \quad d \in \mathbb{R} \\
  z(t) &= \frac{\ell}{4m} \beta(t) + \frac{1}{4m} [(4m - \ell^2) \cos \alpha_0 - \ell \omega_{1,2}] t,
\end{align*}
\]

where \(\beta\) is a non-constant solution of the following ODE:

\[
\beta' + 2md \sin \alpha_0 \cos \beta - 2mc \sin \alpha_0 \sin \beta = \ell \cos \alpha_0 + 2mb \sin^2 \alpha_0 + \omega_{1,2},
\]

and the constants satisfy

\[
e^2 + d^2 = \frac{b}{m} \left\{ (\ell \cos \alpha_0 + \omega_{1,2} - \frac{1}{b}) + mb \sin^2 \alpha_0 \right\}.
\]

**Type II** If \(\beta = \beta_0 = \text{constant}\) and \(\cos \beta_0 \sin \beta_0 \neq 0\), the parametric equations are

\[
\begin{align*}
  x(t) &= x(t) \\
  y(t) &= x(t) \tan \beta_0 + a \\
  z(t) &= \frac{1}{4m} [(4m - \ell^2) \cos \alpha_0 - \ell \omega_{1,2}] t + b, \quad b \in \mathbb{R}
\end{align*}
\]

where \(a = \frac{\omega_{1,2} + \ell \cos \alpha_0}{2m \sin \alpha_0 \cos \beta_0}\) and \(x(t)\) is a solution of the following ODE:

\[
x' = \left( 1 + m[x^2 + (x \tan \beta_0 + a)^2] \right) \sin \alpha_0 \cos \beta_0.
\]

**Type III** If \(\cos \beta_0 \sin \beta_0 = 0\), up to interchenge of \(x\) with \(y\), \(\cos \beta_0 = 0\) and the parametric equations are

\[
\begin{align*}
  x(t) &= x_0 = \frac{\omega_{1,2} + \ell \cos \alpha_0}{2m \sin \alpha_0} \\
  y(t) &= y(t) \\
  z(t) &= \frac{1}{4m} [(4m - \ell^2) \cos \alpha_0 - \ell \omega_{1,2}] t + b, \quad b \in \mathbb{R}
\end{align*}
\]

where \(y(t)\) is a solution of the following ODE:

\[
y' = \pm \left( 1 + m[x_0^2 + y^2] \right) \sin \alpha_0.
\]

Proof. Let \(\gamma(t) = (x(t), y(t), z(t))\) be a curve parametrised by arc length. We shall make use of the Frenet formulas, and we shall take into account Corollary 3.3 and Lemma 4.1. The covariant derivative of the vector field \(T\) given by (12) is

\[
\nabla_T T = \left[ -\beta' \sin \alpha_0 \sin \beta - 2my \sin^2 \alpha_0 \cos \beta \sin \beta \right. \\
\left. + 2mx \sin^2 \alpha_0 \sin^2 \beta + \ell \cos \alpha_0 \sin \alpha_0 \sin \beta \right] E_1 \\
\left. + \left[ \beta' \sin \alpha_0 \cos \beta + 2my \sin^2 \alpha_0 \cos^2 \beta - 2mx \sin^2 \alpha_0 \cos \beta \sin \beta - \ell \cos \alpha_0 \sin \alpha_0 \cos \beta \right] E_2 \right) \\
= kN,
\]
where
\[ k = |\beta' + 2my \sin \alpha_0 \cos \beta - 2mx \sin \alpha_0 \sin \beta - \ell \cos \alpha_0| \sin \alpha_0. \]

We assume that
\[ \omega = \beta' + 2my \sin \alpha_0 \cos \beta - 2mx \sin \alpha_0 \sin \beta - \ell \cos \alpha_0 > 0. \]

Then we have
\[ k = \omega \sin \alpha_0 \]
and
\[ N = -\sin \beta E_1 + \cos \beta E_2. \]

Next,
\[ B = T \times N = -\cos \alpha_0 \cos \beta E_1 - \cos \alpha_0 \sin \beta E_2 + \sin \alpha_0 E_3 \]
and
\[ \nabla_T B = \left[ \beta' \cos \alpha_0 \sin \beta + 2my \sin \alpha_0 \cos \alpha_0 \sin \beta \cos \beta - 2mx \cos \alpha_0 \sin \alpha_0 \sin^2 \beta \right] \\
- \frac{\ell}{2} \cos^2 \alpha_0 \sin \beta + \frac{\ell}{2} \sin^2 \alpha_0 \sin \beta E_1 \\
+ [-\beta' \cos \alpha_0 \cos \beta - 2my \sin \alpha_0 \cos \alpha_0 \cos^2 \beta + 2mx \cos \alpha_0 \sin \alpha_0 \sin \beta \cos \beta] \\
+ \frac{\ell}{2} \cos^2 \alpha_0 \cos \beta - \frac{\ell}{2} \sin^2 \alpha_0 \cos \beta E_2. \]

It follows that the geodesic torsion \( \tau \) of \( \gamma \) is given by
\[ \tau = -\omega \cos \alpha_0 - \frac{\ell}{2}. \]

In order to find the explicit equations for \( \gamma(t) = (x(t), y(t), z(t)) \), we must integrate the system \( d\gamma/dt = T \), that in our case is
\[ \begin{align*}
\frac{x'}{1 + m(x^2 + y^2)} &= \sin \alpha_0 \cos \beta \\
\frac{y'}{1 + m(x^2 + y^2)} &= \sin \alpha_0 \sin \beta \\
z' &= \cos \alpha_0 + \frac{\ell}{2} \sin \alpha_0 (x \sin \beta - y \cos \beta).
\end{align*} \]

We now assume that \( \beta' \neq 0 \), that is we considerer solutions of Type I. Deriveting (18) and taking into account (21) we get
\[ \beta'' = \beta' \frac{2mx' + 2my'}{1 + m(x^2 + y^2)}. \]

By integration of last equation we find
\[ 1 + m(x^2 + y^2) = b\beta', \quad b > 0. \]
Replacing (22) in (21) and integrating we obtain the solution
\[ \begin{align*}
x(t) &= b \sin \alpha_0 \sin \beta(t) + c \\
y(t) &= -b \sin \alpha_0 \cos \beta(t) + d \\
z(t) &= (\cos \alpha_0 + \frac{\ell}{2} \sin^2 \alpha_0) t + \frac{\ell}{2} \int (c \sin \alpha_0 \sin \beta - d \sin \alpha_0 \cos \beta) \, dt.
\end{align*} \]
To determine $\beta$ we replace in (11) the values of $k$, $\tau$ and $B_3$ given in (18), (20) and (19) respectively. This gives

\begin{equation}
\omega^2 + \omega \ell \cos \alpha_0 + (\ell^2 - 4m) \sin^2 \alpha_0 = 0
\end{equation}

Assume that $\delta = \ell^2 + (16m - 5\ell^2) \sin^2 \alpha_0 \geq 0$; then the solutions of (24) are

$$\omega_{1,2} = \frac{-\ell \cos \alpha_0 \pm \sqrt{\delta}}{2},$$

which are always different from zero. Since we have assumed that $\omega$ is a positive constant, we have to choose the positive root of (24). We point out that if $\omega$ is negative we get the same equation (24), thus we keep both solutions of (24).

Replacing in (17) the values of $x$ and $y$ given in (23) we find

$$\beta' - 2m(c \sin \alpha_0 \sin \beta - d \sin \alpha_0 \cos \beta) = \ell \cos \alpha_0 + 2mb \sin^2 \alpha_0 + \omega_{1,2}. $$

Finally, taking into account the integral of the latter equation, the value of $z$ given in (23) becomes the desired ones.

The case of Type II and of Type III can be derived in a similar way. □

Remark 4.3. We point out that the ODE (15) and (16) can be written as the Riccati equation with constant coefficients

\begin{equation}
x' = ax^2 + bx + c, \quad a, b, c \in \mathbb{R},
\end{equation}

which is a separable equation.

The ODE (14) is of type

$$x' = a \cos x + b \sin x + c, \quad a, b, c \in \mathbb{R},$$

which can be reduced to (25).

Remark 4.4. The proper biharmonic curves of Type I, given in (13), lie in the “round cylinder”

\[ S = \{(x, y, z) \in M : (x - c)^2 + (y - d)^2 = b^2 \sin^2 \alpha_0\} \]

and they are geodesics of $S$. The surface $S$ is invariant by translations along the $z$ axis, which are isometries with respect to the Cartan-Vranceanu metric. Similarly, the proper biharmonic curves of Type II and of Type III are geodesics of the “cylinders” $y = x \tan \beta_0 + a$ and $x = x_0$ respectively. For the case $\ell^2 = 4m$, $\ell \neq 0$, i.e. the case of the 3-dimensional sphere, it was proves, in [2], that the proper biharmonic curves are geodesics on the Clifford Torus which is a $SO(2)$-invariant surface of $\mathbb{S}^3$. We can conclude that any proper biharmonic curve of the Cartan-Vranceanu spaces is a geodesic on a surface which is invariant under the action of an 1-parameter group of isometries.

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