This chapter is concerned with the representation and manipulation of complex numbers. Complex numbers pervade this book, underscoring their wide application in the mathematics of the physical sciences. The application of complex numbers to the description of physical systems is left until later chapters and only the basic tools are presented here.

3.1 The need for complex numbers

Although complex numbers occur in many branches of mathematics, they arise most directly out of solving polynomial equations. We examine a specific quadratic equation as an example.

Consider the quadratic equation

$$z^2 - 4z + 5 = 0.$$  \hfill (3.1)

Equation (3.1) has two solutions, $z_1$ and $z_2$, such that

$$(z - z_1)(z - z_2) = 0.$$  \hfill (3.2)

Using the familiar formula for the roots of a quadratic equation, (1.4), the solutions $z_1$ and $z_2$, written in brief as $z_{1,2}$, are

$$z_{1,2} = \frac{4 \pm \sqrt{(-4)^2 - 4(1 \times 5)}}{2}$$

$$= 2 \pm \frac{\sqrt{-4}}{2}. \hfill (3.3)$$

Both solutions contain the square root of a negative number. However, it is not true to say that there are no solutions to the quadratic equation. The *fundamental theorem of algebra* states that a quadratic equation will always have two solutions and these are in fact given by (3.3). The second term on the RHS of (3.3) is called an *imaginary* term since it contains the square root of a negative number;
the first term is called a \textit{real} term. The full solution is the sum of a real term and an imaginary term and is called a \textit{complex number}. A plot of the function $f(z) = z^2 - 4z + 5$ is shown in figure 3.1. It will be seen that the plot does not intersect the $z$-axis, corresponding to the fact that the equation $f(z) = 0$ has no purely real solutions.

The choice of the symbol $z$ for the quadratic variable was not arbitrary; the conventional representation of a complex number is $z$, where $z$ is the sum of a real part $x$ and $i$ times an imaginary part $y$, i.e.

$$
z = x + iy,
$$

where $i$ is used to denote the square root of $-1$. The real part $x$ and the imaginary part $y$ are usually denoted by $\text{Re} \ z$ and $\text{Im} \ z$ respectively. We note at this point that some physical scientists, engineers in particular, use $j$ instead of $i$. However, for consistency, we will use $i$ throughout this book.

In our particular example, $\sqrt{-4} = 2\sqrt{-1} = 2i$, and hence the two solutions of (3.1) are

$$
z_{1,2} = 2 \pm \frac{2i}{2} = 2 \pm i.
$$

Thus here $x = 2$ and $y = \pm 1$.

For compactness a complex number is sometimes written in the form

$$
z = (x, y),
$$

where the components of $z$ may be thought of as coordinates in an $xy$-plot. Such a plot is called an \textit{Argand diagram} and is a common representation of complex numbers; an example is shown in figure 3.2.
Our particular example of a quadratic equation may be generalised readily to polynomials whose highest power (degree) is greater than 2, e.g. cubic equations (degree 3), quartic equations (degree 4) and so on. For a general polynomial \( f(z) \), of degree \( n \), the fundamental theorem of algebra states that the equation \( f(z) = 0 \) will have exactly \( n \) solutions. We will examine cases of higher-degree equations in subsection 3.4.3.

The remainder of this chapter deals with: the algebra and manipulation of complex numbers; their polar representation, which has advantages in many circumstances; complex exponentials and logarithms; the use of complex numbers in finding the roots of polynomial equations; and hyperbolic functions.

### 3.2 Manipulation of complex numbers

This section considers basic complex number manipulation. Some analogy may be drawn with vector manipulation (see chapter 7) but this section stands alone as an introduction.

#### 3.2.1 Addition and subtraction

The addition of two complex numbers, \( z_1 \) and \( z_2 \), in general gives another complex number. The real components and the imaginary components are added separately and in a like manner to the familiar addition of real numbers:

\[
z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2).
\]
3.2 MANIPULATION OF COMPLEX NUMBERS

or in component notation

\[ z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2). \]

The Argand representation of the addition of two complex numbers is shown in figure 3.3.

By straightforward application of the commutativity and associativity of the real and imaginary parts separately, we can show that the addition of complex numbers is itself commutative and associative, i.e.

\[ z_1 + z_2 = z_2 + z_1, \]
\[ z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3. \]

Thus it is immaterial in what order complex numbers are added.

Sum the complex numbers \(1 + 2i, 3 - 4i, -2 + i\).

Summing the real terms we obtain

\[ 1 + 3 - 2 = 2, \]

and summing the imaginary terms we obtain

\[ 2i - 4i + i = -i. \]

Hence

\[ (1 + 2i) + (3 - 4i) + (-2 + i) = 2 - i. \]

The subtraction of complex numbers is very similar to their addition. As in the case of real numbers, if two identical complex numbers are subtracted then the result is zero.
3.2.2 Modulus and argument

The modulus of the complex number \( z \) is denoted by \( |z| \) and is defined as

\[
|z| = \sqrt{x^2 + y^2}.
\]  

(3.4)

Hence the modulus of the complex number is the distance of the corresponding point from the origin in the Argand diagram, as may be seen in figure 3.4.

The argument of the complex number \( z \) is denoted by \( \arg z \) and is defined as

\[
\arg z = \tan^{-1} \left( \frac{y}{x} \right).
\]  

(3.5)

Thus \( \arg z \) is the angle that the line joining the origin to \( z \) on the Argand diagram makes with the positive \( x \)-axis. The anticlockwise direction is taken to be positive by convention. The angle \( \arg z \) is shown in figure 3.4. Account must be taken of the signs of \( x \) and \( y \) individually in determining in which quadrant \( \arg z \) lies. Thus, for example, if \( x \) and \( y \) are both negative then \( \arg z \) lies in the range \(-\pi < \arg z < -\pi/2\) rather than in the first quadrant \((0 < \arg z < \pi/2)\), though both cases give the same value for the ratio of \( y \) to \( x \).

Find the modulus and the argument of the complex number \( z = 2 - 3i \).

Using (3.4), the modulus is given by

\[
|z| = \sqrt{2^2 + (-3)^2} = \sqrt{13}.
\]

Using (3.5), the argument is given by

\[
\arg z = \tan^{-1} \left( \frac{-3}{2} \right).
\]

The two angles whose tangents equal \(-1.5\) are \(-0.9828\) rad and \(2.1588\) rad. Since \( x = 2 \) and \( y = -3 \), \( z \) clearly lies in the fourth quadrant; therefore \( \arg z = -0.9828 \) is the appropriate answer.
3.2 MANIPULATION OF COMPLEX NUMBERS

3.2.3 Multiplication

Complex numbers may be multiplied together and in general give a complex number as the result. The product of two complex numbers $z_1$ and $z_2$ is found by multiplying them out in full and remembering that $i^2 = -1$, i.e.

$$z_1z_2 = (x_1 + iy_1)(x_2 + iy_2) = x_1x_2 + ix_1y_2 + iy_1x_2 + i^2y_1y_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2).$$  \hspace{1cm} (3.6)

Multiply the complex numbers $z_1 = 3 + 2i$ and $z_2 = -1 - 4i$.

By direct multiplication we find

$$z_1z_2 = (3 + 2i)(-1 - 4i) = -3 - 2i - 12i - 8i^2 = 5 - 14i.$$ \hspace{1cm} (3.7)

The multiplication of complex numbers is both commutative and associative, i.e.

$$z_1z_2 = z_2z_1, \hspace{1cm} (3.8)$$

$$(z_1z_2)z_3 = z_1(z_2z_3). \hspace{1cm} (3.9)$$

The product of two complex numbers also has the simple properties

$$|z_1z_2| = |z_1||z_2|, \hspace{1cm} (3.10)$$

$$\arg(z_1z_2) = \arg z_1 + \arg z_2. \hspace{1cm} (3.11)$$

These relations are derived in subsection 3.3.1.

Verify that (3.10) holds for the product of $z_1 = 3 + 2i$ and $z_2 = -1 - 4i$.

From (3.7)

$$|z_1z_2| = |5 - 14i| = \sqrt{5^2 + (-14)^2} = \sqrt{221}.$$  

We also find

$$|z_1| = \sqrt{3^2 + 2^2} = \sqrt{13},$$

$$|z_2| = \sqrt{(-1)^2 + (-4)^2} = \sqrt{17},$$

and hence

$$|z_1||z_2| = \sqrt{13}\sqrt{17} = \sqrt{221} = |z_1z_2|.$$  

We now examine the effect on a complex number $z$ of multiplying it by $\pm 1$ and $\pm i$. These four multipliers have modulus unity and we can see immediately from (3.10) that multiplying $z$ by another complex number of unit modulus gives a product with the same modulus as $z$. We can also see from (3.11) that if
we multiply \( z \) by a complex number, the argument of the product is the sum of the argument of \( z \) and the argument of the multiplier. Hence multiplying \( z \) by unity (which has argument zero) leaves \( z \) unchanged in both modulus and argument, i.e. \( z \) is completely unaltered by the operation. Multiplying by \(-1\) (which has argument \( \pi \)) leads to rotation, through an angle \( \pi \), of the line joining the origin to \( z \) in the Argand diagram. Similarly, multiplication by \( i \) or \(-i\) lead to corresponding rotations of \( \pi/2 \) or \(-\pi/2 \) respectively. This geometrical interpretation of multiplication is shown in figure 3.5.

6 Using the geometrical interpretation of multiplication by \( i \), find the product \( i(1 - i) \).

The complex number \( 1 - i \) has argument \(-\pi/4\) and modulus \( \sqrt{2} \). Thus, using (3.10) and (3.11), its product with \( i \) has argument \(+\pi/4\) and unchanged modulus \( \sqrt{2} \). The complex number with modulus \( \sqrt{2} \) and argument \(+\pi/4\) is \( 1 + i \) and so

\[
i(1 - i) = 1 + i,
\]

as is easily verified by direct multiplication.

The division of two complex numbers is similar to their multiplication but requires the notion of the complex conjugate (see the following subsection) and so discussion is postponed until subsection 3.2.5.

### 3.2.4 Complex conjugate

If \( z \) has the convenient form \( x + iy \) then the complex conjugate, denoted by \( z^* \), may be found simply by changing the sign of the imaginary part, i.e. if \( z = x + iy \) then \( z^* = x - iy \). More generally, we may define the complex conjugate of \( z \) as the (complex) number having the same magnitude as \( z \) that when multiplied by \( z \) leaves a real result, i.e. there is no imaginary component in the product.
3.2 MANIPULATION OF COMPLEX NUMBERS

In the case where \( z \) can be written in the form \( x + iy \) it is easily verified, by direct multiplication of the components, that the product \( zz^* \) gives a real result:

\[
zz^* = (x + iy)(x - iy) = x^2 - ixy + ixy - i^2y^2 = x^2 + y^2 = |z|^2.
\]

Complex conjugation corresponds to a reflection of \( z \) in the real axis of the Argand diagram, as may be seen in figure 3.6.

Find the complex conjugate of \( z = a + 2i + 3b \).

The complex number is written in the standard form

\[
z = a + i(2 + 3b);
\]

then, replacing \( i \) by \( -i \), we obtain

\[
z^* = a - i(2 + 3b).
\]

In some cases, however, it may not be simple to rearrange the expression for \( z \) into the standard form \( x + iy \). Nevertheless, given two complex numbers, \( z_1 \) and \( z_2 \), it is straightforward to show that the complex conjugate of their sum (or difference) is equal to the sum (or difference) of their complex conjugates, i.e. \( (z_1 \pm z_2)^* = z_1^* \pm z_2^* \). Similarly, it may be shown that the complex conjugate of the product (or quotient) of \( z_1 \) and \( z_2 \) is equal to the product (or quotient) of their complex conjugates, i.e. \( (z_1z_2)^* = z_1^*z_2^* \) and \( (z_1/z_2)^* = z_1^*/z_2^* \).

Using these results, it can be deduced that, no matter how complicated the expression, its complex conjugate may always be found by replacing every \( i \) by \( -i \). To apply this rule, however, we must always ensure that all complex parts are first written out in full, so that no \( i \)'s are hidden.
Find the complex conjugate of the complex number \( z = w^{3y+2i} \) where \( w = x + 5i \).

Although we do not discuss complex powers until section 3.5, the simple rule given above still enables us to find the complex conjugate of \( z \).

In this case \( w \) itself contains real and imaginary components and so must be written out in full, i.e.

\[
z = w^{3y+2i} = (x + 5i)^{3y+2i}.
\]

Now we can replace each \( i \) by \(-i\) to obtain

\[
z^* = (x - 5i)^{3y-2i}.
\]

It can be shown that the product \( zz^* \) is real, as required.

The following properties of the complex conjugate are easily proved and others may be derived from them. If \( z = x + iy \) then

\[
(z^*)^* = z, \quad (3.12)
\]

\[
z + z^* = 2 \text{ Re } z = 2x, \quad (3.13)
\]

\[
z - z^* = 2i \text{ Im } z = 2iy, \quad (3.14)
\]

\[
z \frac{z}{z^*} = \left( \frac{x^2 - y^2}{x^2 + y^2} \right) + i \left( \frac{2xy}{x^2 + y^2} \right). \quad (3.15)
\]

The derivation of this last relation relies on the results of the following subsection.

3.2.5 Division

The division of two complex numbers \( z_1 \) and \( z_2 \) bears some similarity to their multiplication. Writing the quotient in component form we obtain

\[
\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2}, \quad (3.16)
\]

In order to separate the real and imaginary components of the quotient, we multiply both numerator and denominator by the complex conjugate of the denominator. By definition, this process will leave the denominator as a real quantity. Equation (3.16) gives

\[
\frac{z_1}{z_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \frac{(x_1x_2 + y_1y_2) + i(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2}
\]

\[
= \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i \frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}.
\]

Hence we have separated the quotient into real and imaginary components, as required.

In the special case where \( z_2 = z_1^* \), so that \( x_2 = x_1 \) and \( y_2 = -y_1 \), the general result reduces to (3.15).
3.3 POLAR REPRESENTATION OF COMPLEX NUMBERS

Express \( z \) in the form \( x + iy \), when
\[
z = \frac{3 - 2i}{-1 + 4i}.
\]

Multiplying numerator and denominator by the complex conjugate of the denominator we obtain
\[
z = \frac{(3 - 2i)(-1 - 4i)}{(-1 + 4i)(-1 - 4i)} = \frac{-11 - 10i}{17} = \frac{-11}{17} - \frac{10}{17}i.
\]

In analogy to (3.10) and (3.11), which describe the multiplication of two complex numbers, the following relations apply to division:
\[
\frac{|z_1|}{|z_2|} = |\frac{z_1}{z_2}|, \quad (3.17)
\]
\[
\arg \left( \frac{z_1}{z_2} \right) = \arg z_1 - \arg z_2. \quad (3.18)
\]

The proof of these relations is left until subsection 3.3.1.

3.3 Polar representation of complex numbers

Although considering a complex number as the sum of a real and an imaginary part is often useful, sometimes the polar representation proves easier to manipulate. This makes use of the complex exponential function, which is defined by
\[
e^z = \exp z \equiv 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots. \quad (3.19)
\]

Strictly speaking it is the function \( \exp z \) that is defined by (3.19). The number \( e \) is the value of \( \exp(1) \), i.e. it is just a number. However, it may be shown that \( e^z \) and \( \exp z \) are equivalent when \( z \) is real and rational and mathematicians then define their equivalence for irrational and complex \( z \). For the purposes of this book we will not concern ourselves further with this mathematical nicety but, rather, assume that (3.19) is valid for all \( z \). We also note that, using (3.19), by multiplying together the appropriate series we may show that (see chapter 20)
\[
e^{z_1}e^{z_2} = e^{z_1 + z_2}, \quad (3.20)
\]
which is analogous to the familiar result for exponentials of real numbers.
From (3.19), it immediately follows that for \( z = i\theta \), \( \theta \) real,

\[
e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \cdots
\]

\[
= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right),
\]

and hence that

\[
e^{i\theta} = \cos \theta + i \sin \theta,
\]

where the last equality follows from the series expansions of trigonometric functions (see subsection 4.6.3). This last relationship is called Euler's equation. It also follows from (3.23) that

\[
e^{n \theta} = \cos n\theta + i \sin n\theta
\]

for all \( n \). From Euler's equation (3.23) and using figure 3.7 we deduce that

\[
re^{i\theta} = r(\cos \theta + i \sin \theta)
\]

\[
= x + iy.
\]

Thus a complex number may be represented in the polar form

\[
z = re^{i\theta}.
\]

Referring again to figure 3.7, we can identify \( r \) with \(|z|\) and \( \theta \) with \( \text{arg } z \). The simplicity of the representation of the modulus and argument is one of the main reasons for using the polar representation. The angle \( \theta \) lies conventionally in the range \(-\pi < \theta \leq \pi\), but, since rotation by \( \theta \) is the same as rotation by \( 2n\pi + \theta \), where \( n \) is any integer,

\[
re^{i\theta} \equiv re^{i(\theta + 2\pi n)}.
\]

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3.3 POLAR REPRESENTATION OF COMPLEX NUMBERS

Figure 3.8  The multiplication of two complex numbers. In this case $r_1$ and $r_2$ are both greater than unity.

The algebra of the polar representation is different from that of the real and imaginary component representation, though, of course, the results are identical. Some operations prove much easier in the polar representation, others much more complicated. The best representation for a particular problem must be determined by the manipulation required.

3.3.1 Multiplication and division in polar form

Multiplication and division in polar form are particularly simple. The product of $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ is given by

$$z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)}.$$  (3.25)

The relations $|z_1 z_2| = |z_1||z_2|$ and $\arg(z_1 z_2) = \arg z_1 + \arg z_2$ follow immediately. An example of the multiplication of two complex numbers is shown in figure 3.8.

Division is equally simple in polar form; the quotient of $z_1$ and $z_2$ is given by

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}.$$  (3.26)

The relations $|z_1/z_2| = |z_1|/|z_2|$ and $\arg(z_1/z_2) = \arg z_1 - \arg z_2$ are again imme-
diately apparent. The division of two complex numbers in polar form is shown in figure 3.9.

### 3.4 de Moivre’s theorem

We now derive an extremely important theorem. Since \((e^{i\theta})^n = e^{in\theta}\), we have

\[
(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta,
\]

(3.27)

where the identity \(e^{in\theta} = \cos n\theta + i \sin n\theta\) follows from the series definition of \(e^{i\theta}\) (see (3.21)). This result is called de Moivre’s theorem and is often used in the manipulation of complex numbers. The theorem is valid for all \(n\) whether real, imaginary or complex.

There are numerous applications of de Moivre’s theorem but this section examines just three: proofs of trigonometric identities; finding the \(n\)th roots of unity; and solving complex equations.

#### 3.4.1 Trigonometric identities

The use of de Moivre’s theorem in finding trigonometric identities is best illustrated by example. We consider the expression of a multiple-angle function in terms of a polynomial in the single-angle function, and its converse.
Express \( \sin 3\theta \) and \( \cos 3\theta \) in terms of powers of \( \cos \theta \) and \( \sin \theta \).

Using de Moivre’s theorem,
\[
\cos 3\theta + i \sin 3\theta = (\cos \theta + i \sin \theta)^3
= (\cos^3 \theta - 3 \cos \theta \sin^2 \theta) + i(3 \sin \theta \cos^2 \theta - \sin^3 \theta). \tag{3.28}
\]
We can equate the real and imaginary coefficients separately, i.e.
\[
\cos 3\theta = \cos^3 \theta - 3 \cos \theta \sin^2 \theta
= 4 \cos^3 \theta - 3 \cos \theta \tag{3.29}
\]
and
\[
\sin 3\theta = 3 \sin \theta \cos^2 \theta - \sin^3 \theta
= 3 \sin \theta - 4 \sin^3 \theta. \tag{3.30}
\]

This method can clearly be applied to finding power expansions of \( \cos n\theta \) and \( \sin n\theta \) for any positive integer \( n \).

The converse process uses the following properties of \( z = e^{i\theta} \),
\[
z^n + \frac{1}{z^n} = 2 \cos n\theta, \tag{3.30}
\]
\[
z^n - \frac{1}{z^n} = 2i \sin n\theta. \tag{3.31}
\]
These equalities follow from simple applications of de Moivre’s theorem, i.e.
\[
z^n + \frac{1}{z^n} = (\cos \theta + i \sin \theta)^n + (\cos \theta + i \sin \theta)^{-n}
= \cos n\theta + i \sin n\theta + \cos(-n\theta) + i \sin(-n\theta)
= \cos n\theta + i \sin n\theta + \cos n\theta - i \sin n\theta
= 2 \cos n\theta
\]
and
\[
z^n - \frac{1}{z^n} = (\cos \theta + i \sin \theta)^n - (\cos \theta + i \sin \theta)^{-n}
= \cos n\theta + i \sin n\theta - \cos n\theta + i \sin n\theta
= 2i \sin n\theta.
\]
In the particular case where \( n = 1 \),
\[
z + \frac{1}{z} = e^{i\theta} + e^{-i\theta} = 2 \cos \theta, \tag{3.32}
\]
\[
z - \frac{1}{z} = e^{i\theta} - e^{-i\theta} = 2i \sin \theta. \tag{3.33}
\]
Find an expression for \( \cos^3 \theta \) in terms of \( \cos 3\theta \) and \( \cos \theta \).

Using (3.32),

\[
\cos^3 \theta = \frac{1}{2^3} \left( z + \frac{1}{z} \right)^3 \\
= \frac{1}{8} \left( z^3 + 3z + \frac{3}{z} + \frac{1}{z^3} \right) \\
= \frac{1}{8} \left( z^3 + \frac{1}{z} \right) + \frac{3}{8} \left( z + \frac{1}{z} \right).
\]

Now using (3.30) and (3.32), we find

\[
\cos^3 \theta = \frac{1}{4} \cos 3\theta + \frac{1}{4} \cos \theta.
\]

This result happens to be a simple rearrangement of (3.29), but cases involving larger values of \( n \) are better handled using this direct method than by rearranging polynomial expansions of multiple-angle functions.

3.4.2 Finding the \( n \)th roots of unity

The equation \( z^2 = 1 \) has the familiar solutions \( z = \pm 1 \). However, now that we have introduced the concept of complex numbers we can solve the general equation \( z^n = 1 \). Recalling the fundamental theorem of algebra, we know that the equation has \( n \) solutions. In order to proceed we rewrite the equation as

\[
z^n = e^{2i\pi k},
\]

where \( k \) is any integer. Now taking the \( n \)th root of each side of the equation we find

\[
z = e^{2i\pi k/n}.
\]

Hence, the solutions of \( z^n = 1 \) are

\[
z_{1,2,\ldots,n} = 1, \quad e^{2i\pi/n}, \quad \ldots, \quad e^{2i(n-1)\pi/n},
\]

corresponding to the values 0, 1, 2, \ldots, \( n - 1 \) for \( k \). Larger integer values of \( k \) do not give new solutions, since the roots already listed are simply cyclically repeated for \( k = n, n + 1, n + 2, \) etc.

Find the solutions to the equation \( z^3 = 1 \).

By applying the above method we find

\[
z = e^{2i\pi/3}.
\]

Hence the three solutions are \( z_1 = e^{i\pi} = 1, \ z_2 = e^{2i\pi/3}, \ z_3 = e^{4i\pi/3} \). We note that, as expected, the next solution, for which \( k = 3 \), gives \( z_4 = e^{6i\pi/3} = 1 = z_1 \), so that there are only three separate solutions.
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Figure 3.10 The solutions of $z^3 = 1$.

Not surprisingly, given that $|z|^3 = |z|^3$ from (3.10), all the roots of unity have unit modulus, i.e. they all lie on a circle in the Argand diagram of unit radius. The three roots are shown in figure 3.10.

The cube roots of unity are often written $1$, $\omega$ and $\omega^2$. The properties $\omega^3 = 1$ and $1 + \omega + \omega^2 = 0$ are easily proved.

3.4.3 Solving polynomial equations

A third application of de Moivre’s theorem is to the solution of polynomial equations. Complex equations in the form of a polynomial relationship must first be solved for $z$ in a similar fashion to the method for finding the roots of real polynomial equations. Then the complex roots of $z$ may be found.

\[\text{Solve the equation } z^6 - z^5 + 4z^4 - 6z^3 + 2z^2 - 8z + 8 = 0.\]

We first factorise to give

\[(z - 2)(z^2 + 4)(z - 1) = 0.\]

Hence $z^3 = 2$ or $z^2 = -4$ or $z = 1$. The solutions to the quadratic equation are $z = \pm 2i$.

to find the complex cube roots, we first write the equation in the form

\[z^3 = 2 = 2e^{2k\pi},\]

where $k$ is any integer. If we now take the cube root, we get

\[z = 2^{1/3}e^{2k\pi/3}.\]
To avoid the duplication of solutions, we use the fact that $-\pi < \arg z \leq \pi$ and find

$$z_1 = 2^{1/3},$$

$$z_2 = 2^{1/3} e^{2\pi i/3} = 2^{1/3} \left( -\frac{1}{2} + \frac{\sqrt{3}}{2}i \right),$$

$$z_3 = 2^{1/3} e^{-2\pi i/3} = 2^{1/3} \left( -\frac{1}{2} - \frac{\sqrt{3}}{2}i \right).$$

The complex numbers $z_1$, $z_2$ and $z_3$, together with $z_4 = 2i$, $z_5 = -2i$ and $z_6 = 1$ are the solutions to the original polynomial equation.

As expected from the fundamental theorem of algebra, we find that the total number of complex roots (six, in this case) is equal to the largest power of $z$ in the polynomial.

A useful result is that the roots of a polynomial with real coefficients occur in conjugate pairs (i.e., if $z_1$ is a root, then $z_1^*$ is a second distinct root, unless $z_1$ is real). This may be proved as follows. Let the polynomial equation of which $z$ is a root be

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0.$$ 

Taking the complex conjugate of this equation,

$$a_n (z^*)^n + a_{n-1} (z^*)^{n-1} + \cdots + a_1 z^* + a_0 = 0.$$ 

But the $a_n$ are real, and so $z^*$ satisfies

$$a_n (z^*)^n + a_{n-1} (z^*)^{n-1} + \cdots + a_1 z^* + a_0 = 0,$$

and is also a root of the original equation.

### 3.5 Complex logarithms and complex powers

The concept of a complex exponential has already been introduced in section 3.3, where it was assumed that the definition of an exponential as a series was valid for complex numbers as well as for real numbers. Similarly we can define the logarithm of a complex number and we can use complex numbers as exponents.

Let us denote the natural logarithm of a complex number $z$ by $w = \ln z$, where the notation $\ln$ will be explained shortly. Thus, $w$ must satisfy

$$z = e^w.$$ 

Using (3.20), we see that

$$z_1 z_2 = e^{w_1} e^{w_2} = e^{w_1 + w_2},$$

and taking logarithms of both sides we find

$$\ln (z_1 z_2) = w_1 + w_2 = \ln z_1 + \ln z_2,$$

which shows that the familiar rule for the logarithm of the product of two real numbers also holds for complex numbers.
We may use (3.34) to investigate further the properties of \( \ln z \). We have already noted that the argument of a complex number is multivalued, i.e. \( \arg z = \theta + 2n\pi \), where \( n \) is any integer. Thus, in polar form, the complex number \( z \) should strictly be written as \( z = re^{i(\theta + 2n\pi)} \).

Taking the logarithm of both sides, and using (3.34), we find
\[
\ln z = \ln r + i(\theta + 2n\pi),
\]
where \( \ln r \) is the natural logarithm of the real positive quantity \( r \) and so is written normally. Thus from (3.35) we see that \( \ln z \) is itself multivalued. To avoid this multivalued behaviour it is conventional to define another function \( \ln z \), the principal value of \( \ln z \), which is obtained from \( \ln z \) by restricting the argument of \( z \) to lie in the range \( -\pi < \theta \leq \pi \).

Evaluate \( \ln(-i) \).

By rewriting \( -i \) as a complex exponential, we find
\[
\ln(-i) = \ln(e^{\pi/2 + 2n\pi}) = i(-\pi/2 + 2n\pi),
\]
where \( n \) is any integer. Hence \( \ln(-i) = -i\pi/2, 3i\pi/2, \ldots \). We note that \( \ln(-i) \), the principal value of \( \ln(-i) \), is given by \( \ln(-i) = -i\pi/2 \).

If \( z \) and \( t \) are both complex numbers then the \( z \)th power of \( t \) is defined by
\[
t^z = e^{z \ln t}.
\]
Since \( \ln t \) is multivalued, so too is this definition.

Simplify the expression \( z = i^{-2i} \).

Firstly we take the logarithm of both sides of the equation to give
\[
\ln z = -2i \ln i.
\]

Now inverting the process we find
\[
e^{\ln z} = z = e^{-2i \ln i}.
\]
We can write \( i = e^{\pi/2 + 2n\pi} \), where \( n \) is any integer, and hence
\[
\ln i = \ln \left[ e^{\pi/2 + 2n\pi} \right] = i \left( \pi/2 + 2n\pi \right).
\]

We can now simplify \( z \) to give
\[
i^{-2i} = e^{-2i \times (\pi/2 + 2n\pi)} = e^{(\pi + 4n)\pi},
\]
which, perhaps surprisingly, is a real quantity rather than a complex one.

Complex powers and the logarithms of complex numbers are discussed further in chapter 20.
We can use the exponential form of a complex number together with de Moivre’s theorem (see section 3.4) to simplify the differentiation of trigonometric functions.

Find the derivative with respect to $x$ of $e^{3x} \cos 4x$.

We could differentiate this function straightforwardly using the product rule (see subsection 2.1.2). However, an alternative method in this case is to use a complex exponential. Let us consider the complex number

$$z = e^{3x}(\cos 4x + i \sin 4x) = e^{3x}e^{i4x} = e^{3x+4ix},$$

where we have used de Moivre’s theorem to rewrite the trigonometric functions as a complex exponential. This complex number has $e^{3x}\cos 4x$ as its real part. Now, differentiating $z$ with respect to $x$ we obtain

$$\frac{dz}{dx} = (3 + 4i)e^{3x+4ix} = (3 + 4i)e^{3x}(\cos 4x + i\sin 4x),$$

where we have again used de Moivre’s theorem. Equating real parts we then find

$$\frac{d}{dx}(e^{3x}\cos 4x) = e^{3x}(3\cos 4x - 4\sin 4x).$$

By equating the imaginary parts of (3.36), we also obtain, as a bonus,

$$\frac{d}{dx}(e^{3x}\sin 4x) = e^{3x}(4\cos 4x + 3\sin 4x).$$

In a similar way the complex exponential can be used to evaluate integrals containing trigonometric and exponential functions.

Evaluate the integral $I = \int e^{ax}\cos bx\,dx$.

Let us consider the integrand as the real part of the complex number

$$e^{ax}(\cos bx + i \sin bx) = e^{ax}e^{ibx} = e^{(a+ib)x},$$

where we use de Moivre’s theorem to rewrite the trigonometric functions as a complex exponential. Integrating we find

$$\int e^{(a+ib)x}\,dx = \frac{e^{(a+ib)x}}{a+ib} + c$$

$$= \frac{(a-ib)e^{(a+ib)x}}{(a-ib)(a+ib)} + c$$

$$= \frac{e^{ax}}{a^2+b^2}(ae^{ibx} - ibe^{ibx}) + c,$$

where the constant of integration $c$ is in general complex. Denoting this constant by $c = c_1 + ic_2$ and equating real parts in (3.37) we obtain

$$I = \int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + c_1,$$

which agrees with result (2.37) found using integration by parts. Equating imaginary parts in (3.37) we obtain, as a bonus,

$$J = \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + c_2.$$
3.7 Hyperbolic functions

The hyperbolic functions are the complex analogues of the trigonometric functions. The analogy may not be immediately apparent and their definitions may appear at first to be somewhat arbitrary. However, careful examination of their properties reveals the purpose of the definitions. For instance, their close relationship with the trigonometric functions, both in their identities and in their calculus, means that many of the familiar properties of trigonometric functions can also be applied to the hyperbolic functions. Further, hyperbolic functions occur regularly, and so giving them special names is a notational convenience.

3.7.1 Definitions

The two fundamental hyperbolic functions are cosh \( x \) and sinh \( x \), which, as their names suggest, are the hyperbolic equivalents of cos \( x \) and sin \( x \). They are defined by the following relations:

\[
\cosh x = \frac{1}{2}(e^x + e^{-x}), \quad (3.38)
\]
\[
\sinh x = \frac{1}{2}(e^x - e^{-x}). \quad (3.39)
\]

Note that \( \cosh x \) is an even function and \( \sinh x \) is an odd function. By analogy with the trigonometric functions, the remaining hyperbolic functions are

\[
\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \quad (3.40)
\]
\[
\sech x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}, \quad (3.41)
\]
\[
\cosech x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}, \quad (3.42)
\]
\[
\coth x = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}. \quad (3.43)
\]

All the hyperbolic functions above have been defined in terms of the real variable \( x \). However, this was simply so that they may be plotted (see figures 3.11–3.13); the definitions are equally valid for any complex number \( z \).

3.7.2 Hyperbolic–trigonometric analogies

In the previous subsections we have alluded to the analogy between trigonometric and hyperbolic functions. Here, we discuss the close relationship between the two groups of functions.

Recalling (3.32) and (3.33) we find

\[
\cos ix = \frac{1}{2}(e^x + e^{-x}),
\]
\[
\sin ix = \frac{1}{2}(e^x - e^{-x}).
\]
Hence, by the definitions given in the previous subsection,

\[
\cosh x = \cos ix, \quad (3.44)
\]
\[
i \sinh x = \sin ix, \quad (3.45)
\]
\[
\cos x = \cosh ix, \quad (3.46)
\]
\[
i \sin x = \sinh ix. \quad (3.47)
\]

These useful equations make the relationship between hyperbolic and trigono-
metric functions transparent. The similarity in their calculus is discussed further in subsection 3.7.6.

### 3.7.3 Identities of hyperbolic functions

The analogies between trigonometric functions and hyperbolic functions having been established, we should not be surprised that all the trigonometric identities also hold for hyperbolic functions, with the following modification. Wherever \( \sin^2 x \) occurs it must be replaced by \( -\sinh^2 x \), and vice versa. Note that this replacement is necessary even if the \( \sin^2 x \) is hidden, e.g. \( \tan^2 x = \sin^2 x / \cos^2 x \) and so must be replaced by \( (-\sinh^2 x / \cosh^2 x) = -\tanh^2 x \).

**Find the hyperbolic identity analogous to** \( \cos^2 x + \sin^2 x = 1 \).

Using the rules stated above \( \cos^2 x \) must be replaced by \( \cosh^2 x \), and \( \sin^2 x \) must be replaced by \( -\sinh^2 x \), and so the identity becomes

\[
\cosh^2 x - \sinh^2 x = 1.
\]

This can be verified by direct substitution, using the definitions of \( \cosh x \) and \( \sinh x \); see (3.38) and (3.39).

Some other identities that can be proved in a similar way are

\[
\begin{align*}
\text{sech}^2 x &= 1 - \tanh^2 x, \\
\text{cosech}^2 x &= \coth^2 x - 1, \\
\sinh 2x &= 2 \sinh x \cosh x, \\
\cosh 2x &= \cosh^2 x + \sinh^2 x.
\end{align*}
\]
3.7.4 Solving hyperbolic equations

When we are presented with a hyperbolic equation to solve, we may proceed by analogy with the solution of trigonometric equations. However, it is almost always easier to express the equation directly in terms of exponentials.

\[ \text{Solve the hyperbolic equation } \cosh x - 5 \sinh x - 5 = 0. \]

Substituting the definitions of the hyperbolic functions we obtain
\[ \frac{1}{2}(e^x + e^{-x}) - \frac{5}{2}(e^x - e^{-x}) - 5 = 0. \]
Rearranging, and then multiplying through by \(-e^x\), gives in turn
\[ -2e^x + 3e^{-x} - 5 = 0 \]
and
\[ 2e^{2x} + 5e^x - 3 = 0. \]
Now we can factorise and solve:
\[ (2e^x - 1)(e^x + 3) = 0. \]
Thus \( e^x = 1/2 \) or \( e^x = -3 \). Hence \( x = -\ln 2 \) or \( x = \ln(-3) \). The interpretation of the logarithm of a negative number has been discussed in section 3.5.

3.7.5 Inverses of hyperbolic functions

Just like trigonometric functions, hyperbolic functions have inverses. If \( y = \cosh x \) then \( x = \cosh^{-1} y \), which serves as a definition of the inverse. By using the fundamental definitions of hyperbolic functions, we can find closed-form expressions for their inverses. This is best illustrated by example.

\[ \text{Find a closed-form expression for the inverse hyperbolic function } y = \sinh^{-1} x. \]

First we write \( x \) as a function of \( y \), i.e.
\[ y = \sinh^{-1} x \implies x = \sinh y. \]
Now, since \( \cosh y = \frac{1}{2}(e^y + e^{-y}) \) and \( \sinh y = \frac{1}{2}(e^y - e^{-y}) \),
\[ e^y = \cosh y + \sinh y = \sqrt{1 + \sinh^2 y} + \sinh y \]
\[ e^y = \sqrt{1 + x^2 + x}, \]
and hence
\[ y = \ln(\sqrt{1 + x^2} + x). \]

In a similar fashion it can be shown that
\[ \cosh^{-1} x = \ln(\sqrt{x^2 - 1} + x). \]
3.7 HYPERBOLIC FUNCTIONS

Figure 3.14 Graphs of \( \cosh^{-1} x \) and \( \text{sech}^{-1} x \).

Find a closed-form expression for the inverse hyperbolic function \( y = \tanh^{-1} x \).

First we write \( x \) as a function of \( y \), i.e.

\[
y = \tanh^{-1} x \quad \Rightarrow \quad x = \tanh y.
\]

Now, using the definition of \( \tanh y \) and rearranging, we find

\[
x = \frac{e^y - e^{-y}}{e^y + e^{-y}} \quad \Rightarrow \quad (x + 1)e^{-y} = (1 - x)e^y.
\]

Thus, it follows that

\[
e^{2y} = \frac{1 + x}{1 - x} \quad \Rightarrow \quad e^y = \sqrt{\frac{1 + x}{1 - x}},
\]

\[
y = \ln \sqrt{\frac{1 + x}{1 - x}}.
\]

\[
\tanh^{-1} x = \frac{1}{2} \ln \left( \frac{1 + x}{1 - x} \right).
\]

Graphs of the inverse hyperbolic functions are given in figures 3.14–3.16.

3.7.6 Calculus of hyperbolic functions

Just as the identities of hyperbolic functions closely follow those of their trigonometric counterparts, so their calculus is similar. The derivatives of the two basic
Complex numbers and hyperbolic functions

\[
\sinh^{-1} x = \frac{1}{2} \ln \left( x + \sqrt{x^2 + 1} \right)
\]

\[
\cosech^{-1} x = \ln \left( \frac{1}{x} + \sqrt{\frac{1}{x^2} + 1} \right)
\]

\[
\coth^{-1} x = \frac{1}{2} \ln \left( \frac{x + 1}{x - 1} \right)
\]

\[
\tanh^{-1} x = \frac{1}{2} \ln \left( \frac{1 + x}{1 - x} \right)
\]

Figure 3.15 Graphs of \( \sinh^{-1} x \) and \( \cosech^{-1} x \).

Figure 3.16 Graphs of \( \tanh^{-1} x \) and \( \coth^{-1} x \).

Hyperbolic functions are

\[
\frac{d}{dx} (\cosh x) = \sinh x, \quad (3.52)
\]

\[
\frac{d}{dx} (\sinh x) = \cosh x. \quad (3.53)
\]

These may be deduced by considering the definitions.
Verify the relation \((d/dx) \cosh x = \sinh x\).

Using the definition of \(\cosh x\),

\[
\cosh x = \frac{1}{2} (e^x + e^{-x}),
\]

and differentiating directly, we find

\[
\frac{d}{dx} (\cosh x) = \frac{1}{2} (e^x - e^{-x}) = \sinh x.
\]

Clearly the integrals of the fundamental hyperbolic functions are also defined by these relations. The derivatives of the remaining hyperbolic functions can be derived by product differentiation and are presented below only for completeness.

\[
\frac{d}{dx} (\tanh x) = \text{sech}^2 x, \quad (3.54)
\]

\[
\frac{d}{dx} (\text{sech} x) = -\text{sech} x \tanh x, \quad (3.55)
\]

\[
\frac{d}{dx} (\text{cosech} x) = -\text{cosech} x \coth x, \quad (3.56)
\]

\[
\frac{d}{dx} (\coth x) = -\text{cosech}^2 x. \quad (3.57)
\]

The inverse hyperbolic functions also have derivatives, which are given by the following:

\[
\frac{d}{dx} \left( \cosh^{-1} \frac{x}{a} \right) = \frac{1}{\sqrt{x^2 - a^2}}, \quad (3.58)
\]

\[
\frac{d}{dx} \left( \sinh^{-1} \frac{x}{a} \right) = \frac{1}{\sqrt{x^2 + a^2}}, \quad (3.59)
\]

\[
\frac{d}{dx} \left( \tanh^{-1} \frac{x}{a} \right) = \frac{a}{a^2 - x^2}, \quad \text{for } x^2 < a^2, \quad (3.60)
\]

\[
\frac{d}{dx} \left( \coth^{-1} \frac{x}{a} \right) = \frac{-a}{x^2 - a^2}, \quad \text{for } x^2 > a^2. \quad (3.61)
\]

These may be derived from the logarithmic form of the inverse (see subsection 3.7.5).
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Evaluate \((d/dx)\sinh^{-1}x\) using the logarithmic form of the inverse.

From the results of section 3.7.5,

\[
\frac{d}{dx} (\sinh^{-1} x) = \frac{d}{dx} \left[ \ln \left( x + \sqrt{x^2 + 1} \right) \right]
\]

\[
= \frac{1}{x + \sqrt{x^2 + 1}} \left( 1 + \frac{x}{\sqrt{x^2 + 1}} \right)
\]

\[
= \frac{1}{x + \sqrt{x^2 + 1}} \left( \frac{x}{\sqrt{x^2 + 1}} + 1 \right)
\]

\[
= \frac{1}{\sqrt{x^2 + 1}}.
\]

3.8 Exercises

3.1 Two complex numbers \(z\) and \(w\) are given by \(z = 3 + 4i\) and \(w = 2 - i\). On an Argand diagram plot
(a) \(z + w\), (b) \(w - z\), (c) \(zw\), (d) \(z/w\),
(e) \(z^*w + w^*z\), (f) \(w^2\), (g) \(\ln z\), (h) \((1 + z + w)^{1/2}\).

3.2 By considering the real and imaginary parts of the product \(e^{\theta+i\phi}\) prove the standard formulae for \(\cos(\theta + \phi)\) and \(\sin(\theta + \phi)\).

3.3 By writing \(\pi/12 = (\pi/3) - (\pi/4)\) and considering \(e^{i\pi/12}\), evaluate \(\cot(\pi/12)\).

3.4 Find the locus in the complex \(z\)-plane of points that satisfy the following equations.
(a) \(z - c = \rho \left( \frac{1 + it}{1 - it} \right)\), where \(c\) is complex, \(\rho\) is real and \(t\) is a real parameter that varies in the range \(-\infty < t < \infty\).
(b) \(z = a + bt + ct^2\), in which \(t\) is a real parameter and \(a, b, c\) are complex numbers with \(b/c\) real.

3.5 Evaluate
(a) \(\text{Re}(\exp 2iz)\), (b) \(\text{Im}(\cosh^2 z)\), (c) \((-1 + \sqrt{3}i)^{1/2}\),
(d) \(\exp(i/2)\), (e) \(\exp(i\cdot 2\pi)\), (f) \(\text{Im}(2i^{1/3})\), (g) \(i\cdot 2\), (h) \(\ln((\sqrt{3} + i)^3)\).

3.6 Find the equations in terms of \(x\) and \(y\) of the sets of points in the Argand diagram that satisfy the following:
(a) \(\text{Re} z^2 = \text{Im} z^2\);
(b) \(\text{Im}(z^2)/z^2 = -i\);
(c) \(\arg(z/(z - 1)) = \pi/2\).

3.7 Show that the locus of all points \(z = x + iy\) in the complex plane that satisfy
\[
|z - ia| = \lambda |z + ia|, \quad \lambda > 0,
\]
is a circle of radius \(2\lambda/(1 - \lambda^2)|a|\) centred on the point \(z = ia[(1 + \lambda^2)/(1 - \lambda^2)]\). Sketch the circles for a few typical values of \(\lambda\), including \(\lambda < 1\), \(\lambda > 1\) and \(\lambda = 1\).

3.8 The two sets of points \(z = a, z = b, z = c\), and \(z = A, z = B, z = C\) are the corners of two similar triangles in the Argand diagram. Express in terms of \(a, b, \ldots, C\).
3.8 EXERCISES

(a) the equalities of corresponding angles, and
(b) the constant ratio of corresponding sides,
in the two triangles.

By noting that any complex quantity can be expressed as
\[ z = |z| \exp(i \arg z), \]
deduce that
\[ a(B - C) + b(C - A) + c(A - B) = 0. \]

3.9 For the real constant \( a \) find the loci of all points \( z = x + iy \) in the complex plane that satisfy

\[ (a) \quad \text{Re} \left\{ \ln \left( \frac{z - ia}{z + ia} \right) \right\} = c, \quad c > 0, \]
\[ (b) \quad \text{Im} \left\{ \ln \left( \frac{z - ia}{z + ia} \right) \right\} = k, \quad 0 \leq k \leq \pi/2. \]

Identify the two families of curves and verify that in case (b) all curves pass through the two points \( \pm ia \).

3.10 The most general type of transformation between one Argand diagram, in the \( z \)-plane, and another, in the \( Z \)-plane, that gives one and only one value of \( Z \) for each value of \( z \) (and conversely) is known as the \textit{general bilinear transformation} and takes the form
\[ z = \frac{aZ + b}{cZ + d}. \]

(a) Confirm that the transformation from the \( Z \)-plane to the \( z \)-plane is also a general bilinear transformation.

(b) Recalling that the equation of a circle can be written in the form
\[ \frac{|z - z_1|}{|z - z_2|} = \lambda, \quad \lambda \neq 1, \]
show that the general bilinear transformation transforms circles into circles (or straight lines). What is the condition that \( z_1, z_2 \) and \( \lambda \) must satisfy if the transformed circle is to be a straight line?

3.11 Sketch the parts of the Argand diagram in which

\[ (a) \quad \text{Re} \, z^2 < 0, \, |z^{1/2}| \leq 2, \]
\[ (b) \quad 0 \leq \arg z^3 \leq \pi/2, \]
\[ (c) \quad |\exp z^4| \to 0 \text{ as } |z| \to \infty. \]

What is the area of the region in which all three conditions are satisfied?

3.12 Denote the \( n \)th roots of unity by \( 1, \, \omega_n, \, \omega_n^2, \ldots, \, \omega_n^{n-1} \).

(a) Prove that
\[ (i) \quad \sum_{r=0}^{n-1} \omega_n^r = 0, \quad (ii) \quad \prod_{r=0}^{n-1} \omega_n^r = (-1)^{n+1}. \]

(b) Express \( x^2 + y^2 + z^2 - yz - zx - xy \) as the product of two factors, each linear in \( x, \, y \) and \( z \), with coefficients dependent on the third roots of unity (and those of the \( x \) terms arbitrarily taken as real).
3.13 Prove that $x^{2m+1} - a^{2m+1}$, where $m$ is an integer $\geq 1$, can be written as

$$x^{2m+1} - a^{2m+1} = (x - a) \prod_{r=1}^{m} \left[ x^{2} - 2ax \cos \left( \frac{2\pi r}{2m+1} \right) + a^{2} \right].$$

3.14 The complex position vectors of two parallel interacting equal fluid vortices moving with their axes of rotation always perpendicular to the z-plane are $z_{1}$ and $z_{2}$. The equations governing their motions are

$$dz_{1}^{*} / dt = -i z_{1} - z_{2}, \quad dz_{2}^{*} / dt = -i z_{2} - z_{1}.$$ 

Deduce that (a) $z_{1} + z_{2}$, (b) $|z_{1} - z_{2}|$ and (c) $|z_{1}|^{2} + |z_{2}|^{2}$ are all constant in time, and hence describe the motion geometrically.

3.15 Solve the equation

$$z^{7} - 4z^{6} + 6z^{5} - 6z^{4} + 6z^{3} - 12z^{2} + 8z + 4 = 0,$$

(a) by examining the effect of setting $z^{3} = 2$, and (b) by factorising and using the binomial expansion of $(z + a)^{3}$.

Plot the seven roots of the equation on an Argand plot, exemplifying that complex roots of a polynomial equation always occur in conjugate pairs if the polynomial has real coefficients.

3.16 The polynomial $f(z)$ is defined by

$$f(z) = z^{5} - 6z^{4} + 15z^{3} - 34z^{2} + 36z - 48.$$ 

(a) Show that the equation $f(z) = 0$ has roots of the form $z = \lambda i$ where $\lambda$ is real, and hence factorize $f(z)$.

(b) Show further that the cubic factor of $f(z)$ can be written in the form $(z + a)^{3} + b$, where $a$ and $b$ are real, and hence solve the equation $f(z) = 0$ completely.

3.17 The binomial expansion of $(1 + x)^{n}$, discussed in chapter 1, can be written for a positive integer $n$ as

$$(1 + x)^{n} = \sum_{r=0}^{n} \binom{n}{r} x^{r},$$

where $\binom{n}{r} = n! / [r!(n-r)!]$.

(a) Use de Moivre’s theorem to show that the sum

$$S_{1}(n) = \binom{n}{0} - \binom{n}{2} + \binom{n}{4} - \cdots + (-1)^{m} \binom{n}{2m}, \quad n - 1 \leq 2m \leq n,$$

has the value $2^{n/2} \cos(n\pi/4)$.

(b) Derive a similar result for the sum

$$S_{2}(n) = \binom{n}{1} - \binom{n}{3} + \binom{n}{5} - \cdots + (-1)^{m} \binom{n}{2m+1}, \quad n - 1 \leq 2m + 1 \leq n,$$

and verify it for the cases $n = 6, 7$ and 8.

3.18 By considering $(1 + \exp i\theta)^{n}$, prove that

$$\sum_{r=0}^{n} \binom{n}{r} \cos r\theta = 2^{n} \cos(n\theta/2) \cos(n\theta/2),$$

$$\sum_{r=0}^{n} \binom{n}{r} \sin r\theta = 2^{n} \cos(n\theta/2) \sin(n\theta/2),$$

where $\binom{n}{r} = n! / [r!(n-r)!]$. 

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3.19 Use de Moivre’s theorem with \( n = 4 \) to prove that
\[
\cos 4\theta = 8\cos^4 \theta - 8\cos^2 \theta + 1,
\]
and deduce that
\[
\cos \frac{\pi}{8} = \left( \frac{2 + \sqrt{2}}{4} \right)^{1/2}.
\]

3.20 Express \( \sin^4 \theta \) entirely in terms of the trigonometric functions of multiple angles and deduce that its average value over a complete cycle is \( \frac{1}{8} \).

3.21 Use de Moivre’s theorem to prove that
\[
\tan 5\theta = \frac{5\tan \theta - 10\tan^3 \theta + 5\tan \theta}{5\tan^2 \theta - 10\tan^2 \theta + 1},
\]
where \( t = \tan \theta \). Deduce the values of \( \tan(n\pi/10) \) for \( n = 1, 2, 3, 4 \).

3.22 (a) Prove that
\[
\cosh x - \cosh y = 2 \sinh \left( \frac{x+y}{2} \right) \sinh \left( \frac{x-y}{2} \right).
\]
(b) Prove that, if \( y = \sinh^{-1} x \),
\[
(x^2 + 1) \frac{d^2 y}{dx^2} + x \frac{dy}{dx} = 0.
\]

3.23 Determine the conditions under which the equation
\[
a \cosh x + b \sinh x = c, \quad c > 0,
\]
has zero, one, or two real solutions for \( x \). What is the solution if \( a^2 = c^2 + b^2 \)?

3.24 (a) Solve \( \cosh x = \sinh x + 2 \text{sech} x \).
(b) Show that the real solution \( x \) of \( \tanh x = \cosech x \) can be written in the form \( x = \ln(u + \sqrt{u}) \). Find an explicit value for \( u \).
(c) Evaluate \( \tanh x \) when \( x \) is the real solution of \( \cosh 2x = 2 \cosh x \).

3.25 Express \( \sinh^4 x \) in terms of hyperbolic cosines of multiples of \( x \), and hence solve
\[
2 \cosh 4x - 8 \cosh 2x + 5 = 0.
\]

3.26 In the theory of special relativity, the relationship between the position and time coordinates of an event as measured in two frames of reference that have parallel \( x \)-axes can be expressed in terms of hyperbolic functions. If the coordinates are \( x \) and \( t \) in one frame and \( x' \) and \( t' \) in the other then the relationship take the form
\[
x' = x \cosh \phi - ct \sinh \phi,
\]
\[
c't' = -x \sinh \phi + ct \cosh \phi.
\]
Express \( x \) and \( ct \) in terms of \( x' \), \( ct' \) and \( \phi \) and show that
\[
x^2 - (ct)^2 = (x')^2 - (ct')^2.
\]

3.27 A closed barrel has as its curved surface that obtained by rotating about the \( x \)-axis the part of the curve
\[
y = a[2 - \cosh(x/a)]
\]
lying in the range \(-b \leq x \leq b\). Show that the total surface area \( A \) of the barrel is given by
\[
A = \pi a[9a - 8a \exp(-b/a) + a \exp(-2b/a) - 2b].
\]
3.13 Denoting $e^{2\pi i/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$.

3.14 (b) Differentiate $(\tan(w))^2$.

3.28 The principal value of the logarithmic function of a complex variable is defined to have its argument in the range $-\pi < \arg z \leq \pi$. By writing $z = \tan w$ in terms of exponentials show that

$$\tan^{-1} z = \frac{1}{2i} \ln \left( \frac{1 + iz}{1 - iz} \right).$$

Use this result to evaluate

$$\tan^{-1} \left( \frac{2\sqrt{3} - 3i}{7} \right).$$

3.9 Hints and answers

3.1 (a) $5 + 3i$; (b) $-1 - 5i$; (c) $10 + 5i$; (d) $2/5 + 11i/5$; (e) $4$; (f) $3 - 4i$;

(g) $5 + i[\tan^{-1}(4/3) + 2n\pi]$; (b) $\pm(2.521 + 0.595i)$.

3.3 $2 + \sqrt{3}$.

3.4 (a) Set $t = \tan \theta$ with $-\pi/2 < \theta < \pi/2$. The equation becomes $z - c = pe^{i\theta}$.

The locus is a circle, centre $c$, radius $p$.

(b) Eliminate the $t$ term between $x$ and $y$. Note that the coefficient of $t$ is proportional to $\Im(b/c)$. The locus is a straight line $(\Im(k)[x - \Re(a)] = (\Re(k))[y - \Im(a)])$, where $k = b$ or $c$.

3.5 (a) $\exp(-2y)\cos 2x$; (b) $(\sin 2y)\sin(2x)/2$; (c) $\sqrt{2}\exp(\pi i/3)$ or $\sqrt{2}\exp(4\pi i/3)$;

(d) $\exp(1/\sqrt{2})$ or $\exp(-1/\sqrt{2})$; (c) $0.540 - 0.841i$; (f) $8\sin(\ln 2) = 5.11$;

(g) $\exp(-\pi/2 - 2\pi n)$; (b) $\ln 8 + i(2n + 1/2)\pi$.

3.6 (a) $y = (\pm\sqrt{2} - 1)x$; (b) $x = \pm y$; (c) the half of the circle $(x - \frac{1}{2})^2 + y^2 = \frac{1}{4}$ that lies in $y < 0$.

3.7 Starting from $|x + iy - ia| = \lambda|x + iy + ia|$, show that the coefficients of $x$ and $y$ are equal, and write the equation in the form $a^2 + (y-z)^2 = r^2$.

3.8 (a) $\arg([b - a]/[c - a]) = \arg([B - A]/[C - A])$.

(b) $|[b - a]|/|c - a| = \|(B - A)/[(C - A)]$.

3.9 (a) Circles enclosing $z = -ia$, with $\lambda = \exp c > 1$.

(b) The condition is that $\arg((z - ia)/(z + ia)) = k$. This can be rearranged to give $a(z + z') = k(a^2 - |z|^2)$, which becomes in $x, y$ coordinates the equation of a circle with centre $(-a/k, 0)$ and radius $a(1 + k^{-2})/2$.

3.10 (a) $Z = (-(dz + b)/(cz - a)$.

(b) $|(Z - Z_1)/(Z - Z_2)| = A$, with $Z_{1,2}$ given by setting $z = z_{1,2}$ in the result in (a); $|z - cz| = \lambda|a - cz|$

3.11 All three conditions are satisfied in $3\pi/2 \leq \theta \leq 7\pi/4$, $|z| \leq 4$; area $= 2\pi$.

3.12 (a) Express $\omega^3 = 1$ as a product of factors like $(\omega - \omega_0^*)$ and examine the coefficients of (i) $\omega^{n+1}$ and (ii) $\omega^n$.

(b) $\lambda = \exp a$ or $\omega_0z (x + \omega_0^2z)$.

3.13 Denoting $\exp(2\pi i/(2m + 1))$ by $\Omega$, express $x^{2m+1} - a^{2m+1}$ as a product of factors like $(x - a\Omega^p)$ and then combine those containing $\Omega^p$ and $\Omega^{2m+1-p}$. Use the fact that $\Omega^{2m+1} = 1$.

3.14 (b) Differentiate $(z_1 - z_2)(z_1^* - z_2^*)$. (c) Write $2|z_1|^2 + 2|z_2|^2$ as $|z_1 + z_2|^2 - |z_1 - z_2|^2$.

Circular motion about a fixed point with the vortices at the opposite ends of a diameter.

3.15 The roots are $2^{1/3}\exp(2\pi ni/3)$ for $n = 0, 1, 2; 1 + 3^{1/4}; 1 + 3^{1/4}$.

3.16 (a) The vanishing of the real and imaginary parts of $f(\lambda i)$ requires $(\lambda^2 = 3$ or $\frac{\lambda}{2})$

and $(\lambda^2 = 0$ or $3$ or $12$); hence $\lambda^2 = 3$ and $f(z) = (z^3 + 3)(z^3 - 6z^2 + 12z - 16)$.

(b) $a = -2, b = -8$. The roots are $\pm i\sqrt{3}$, $4, 1 + i\sqrt{3}$.
3.17 (b) \( S_2(n) = 2^{n/2} \sin(n\pi/4) \). \( S_2(6) = -8 \), \( S_2(7) = -8 \), \( S_2(8) = 0 \).

3.18 Write \( 1 + \cos \theta \) and \( \sin \theta \) in terms of \( \theta/2 \).

3.20 \( (\cos 4\theta)/8 - (\cos 2\theta)/2 + 3/8 \).

3.21 Show that \( \cos 5\theta = 16c^5 - 20c^3 + 5c \), where \( c = \cos \theta \), and correspondingly for \( \sin 5\theta \). Use \( \cos^{-2} \theta = 1 + \tan^2 \theta \). The four required values are \([ (5 - \sqrt{20})/5 ]^{1/2}, (5 - \sqrt{20})^{1/2}, [(5 + \sqrt{20})/5]^{1/2}, (5 + \sqrt{20})^{1/2} \).

3.23 Reality of the root(s) requires \( c^2 + b^2 \geq a^2 \) and \( a + b > 0 \). With these conditions, there are two roots if \( a^2 > b^2 \), but only one if \( b^2 > a^2 \).

For \( a^2 = c^2 + b^2 \), \( x = \frac{1}{2} \ln[(a - b)/(a + b)] \).

3.24 (a) \( \ln(1/\sqrt{3}) \); (b) \( (1 + \sqrt{3})/2 \); (c) \( \pm(12)^{1/4}/(\sqrt{3} + 1) \).

3.25 Reduce the equation to \( 16 \sinh^2 x = 1 \), yielding \( x = \pm0.481 \).

3.26 The same expressions but with \( \phi \) replaced by \( -\phi \) are obtained.

3.27 Show that \( ds = (\cosh x/a) \, dx \);

\[ \text{curved surface area} = \pi a^2 [8 \sinh(b/a) - \sinh(2b/a)] - 2\pi ab. \]

3.28 \( \pi/6 - i \ln \sqrt{2} \).