The Philosophy of Science in a European Perspective

Hanne Andersen · Dennis Dieks
Wenceslao J. Gonzalez · Thomas Uebel
Gregory Wheeler *Editors*

New Challenges to Philosophy of Science

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Hector Freytes, Antonio Ledda, Giuseppe Sergioli and Roberto Giuntini

Probabilistic Logics in Quantum Computation

Abstract

The quantum computation process may be summarized as follows: first an initial state of a physical system is provided as the input. Then, it evolves according to the elementary operations (quantum gates) that are performed on it. Finally, the access to the information content of the resulting state is possible via the measurement operation that provides one of the possible results. In this note we describe probabilistic-type semantics for propositional logics designed to describe effective procedure based on measurement processes.

1. Introduction

Probabilistic logics are conceived to represent the fact that a valid argument is one in which it is not possible for the probability-values of all the premises to be high, while the probability-value of the conclusion is not. More generally, the interest of these logics is to study the propagation of probability-values from the premises to the conclusion of a valid argument. If the premises of a valid argument are all certain, then so is the conclusion. *Probability logics* is the name that Adams [1] proposes for the formal study of the transmission (or lack thereof) of probability-values through valid inferences. Clearly, those basic ideas can be generalized. In fact, alternative axiomatizations of probability defined over event structures different from the usual Boolean σ-algebras bring out alternative logics. This is, in fact, the case of quantum probability [21], or the investigation of states over orthostructures [8, 12, 9]. In this note we describe possible probabilistic semantics arising from probability-values of quantum measurement. The paper is structured as follows: in Section 2, we briefly recall some required basic notions in order to make the paper self-contained. Sections 3 and 4 describe the idea of probabilistic-type logics for pure and mixed states, respectively. Finally, Section 5 outlines possible connections between probabilistic semantics arising from quantum measurement probabilities and fuzzy logic.

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2. Preliminary notions

The notion of *state of a physical system* is familiar in classical mechanics, where it is related to the initial conditions (the initial values of position and momentum) which determine the solutions of the equation of motion of the system. For any value of time, the state is represented by a point in the phase space. In the quantum framework, the description of a state is substantially modified.

Before giving the definition of quantum state, we introduce the concept of *maximal quantum test*. Suppose that we want to observe the properties of a quantum system that can possibly take n different values. If the test you devise allows to distinguish among n possibilities, we say that it is a maximal test. A n-outcome measurement of those properties implements a maximal test. A test that gives only partial information is said to be a partial test. If a quantum system is prepared in such a way that one can devise a maximal test yielding with certainty a particular outcome, then we say that the quantum system is in a *pure state*. The pure state of a quantum system is described by a unit vector in a Hilbert space, and it is denoted by $|\varphi\rangle$ in Dirac notation. If the maximal test for a pure state has n possible outcomes, the state is described by a vector $|\varphi\rangle$ in a n-dimensional Hilbert space. Any orthornormal basis represents a realisable maximal test. Suppose that we have a large number of similarly prepared systems, called an ensemble, and we test for the values of different measurable quantities like, e.g., spin etc. In general, we postulate that, for an ensemble in an arbitrary state, it is always possible to devise a test that yields the n outcomes corresponding to an orthonormal basis with definite probabilities. If the system is prepared in a state $|\varphi\rangle$, and a maximal test corresponding to a basis $\{|e_1\rangle, \ldots, |e_n\rangle\}$ is performed, the probability that the outcome will correspond to $|e_i\rangle$ is given by $p_i(|\varphi\rangle) = |\langle e_i|\varphi\rangle|^2$.

The idea of quantum computation was introduced in 1982 by Richard Feynmann and remained primarily of theoretical interest until developments such as, e.g., Shor's factorization algorithm, that triggered a vast domain of research. In a classical computer, the information is encoded in a series of bits, that are manipulated by logical gates, arranged in a suitable sequence to produce the output. Standard quantum computing is based on quantum systems described by finite dimensional Hilbert spaces, specially $\mathbb{C}^2$, the two-dimensional space of the *qbit*. Similarly to the classical computing case, we can introduce and study the behavior of a number of *quantum logical gates* (hereafter quantum gates for short) acting on qbits. Quantum computing can simulate any computation perfomed by a classical system; however, one of the main advantages of quantum computation and quantum algorithms is that they can speed up the processes.

The standard orthonormal basis $\{|0\rangle, |1\rangle\}$ of $\mathbb{C}^2$ (where $|0\rangle = (1, 0)$ and $|1\rangle = (0, 1)$) is called the *logical* (or *computational*) *basis*. Thus, pure states $|\varphi\rangle$ in $\mathbb{C}^2$ are coherent superpositions of the basis vectors with complex coefficients following the Born rule.

Any qbit $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$ may be regarded as a piece of information, where the number $|c_0|^2$ corresponds to the probability-value that the information described by the basic state $|0\rangle$ is false; while $|c_1|^2$ corresponds to the probability-value that the information described by the basic state $|1\rangle$ is true. The two basis-elements $|0\rangle$ and $|1\rangle$ are usually taken as encoding the classical bit-values 0 and 1, respectively. By these means, a probability value is assigned to a qbit as follows:

Definition 0.0.1 Let $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$ be a qbit. Then its *probability value* is $p(|\psi\rangle) = |c_1|^2$

Generalizing for a positive integer n, *n-qbits* are represented by unit vectors in the 2^n-dimensional complex Hilbert space $\otimes^n\mathbb{C}^2$. A special basis, called the 2^n-*computational basis*, is chosen for $\otimes^n\mathbb{C}^2$. More precisely, it consists of the 2^n orthogonal states $|\iota\rangle$, with $0 \leq \iota \leq 2^n$, where ι is in binary representation, and $|\iota\rangle$ can be seen as the tensor product of the states $|\iota_1\rangle \otimes |\iota_2\rangle \otimes \ldots \otimes |\iota_n\rangle$, with $\iota_j \in \{0, 1\}$. In this case $\otimes^n|1\rangle = (0, 0\ldots 0, 1)$ is the n-qbit, in the computational basis, encoding the classical bit 1 in $\otimes^n\mathbb{C}^2$.

A n-qbit $|\psi\rangle \in \otimes^n\mathbb{C}^2$ is a superposition of the basis vectors $|\psi\rangle = \sum_{\iota=1}^{2^n} c_\iota|\iota\rangle$, with $\sum_{\iota=1}^{2^n} |c_\iota|^2 = 1$, and the probability assigned to $|\psi\rangle$ is $|c_{0,0\ldots 0,1}|^2$.

In the usual representation of quantum computational processes, a quantum circuit is identified with an appropriate composition of *quantum gates*, i.e. unitary operators acting on pure states of a convenient Hilbert space $\otimes^n\mathbb{C}^2$ [20]. Consequently, quantum gates represent time reversible evolutions of pure states of the system.

3. Probabilistic-type logic for qbits

Let X be a nonempty set, whose elements are referred as propositional variables, and $\mathfrak{F}$ be a set of connectives each of them with its respective arity. Let $\mathcal{L}_{\mathfrak{F}}$ be the propositional language from X and $\mathfrak{F}$. A probabilistic logic for qbits may be introduced as a logic $\langle\mathcal{L}_{\mathfrak{F}}, \models\rangle$, where the propositional variables are interpreted as n-qbits in a given Hilbert space $\otimes^n\mathbb{C}^2$, and the connectives are naturally interpreted as unitary operators acting on pure states in $\otimes^n\mathbb{C}^2$. More precisely, let $\mathcal{Q}(n)$ be the set of n-qbits in $\otimes^n\mathbb{C}^2$ and, if $f \in \mathfrak{F}$, let U_f denote the unitary operator associated to f in $\otimes^n\mathbb{C}^2$.

An interpretation of $\mathcal{L}_{\mathfrak{F}}$ in $\mathcal{Q}(n)$ is any function $e : \mathcal{L}_{\mathfrak{F}} \to \mathcal{Q}(n)$ such that, for each $f \in \mathfrak{F}$ with arity k,

$$e(f(x_1, \ldots x_k)) = U_f(e(x_1)\ldots e(x_k)).$$

To define a semantic consequence relation $\models$ from the probability assignment, another step is required: the notion of evaluation. An *evaluation* is any function $v : \mathcal{L}_{\mathfrak{F}} \to [0, 1]$ such that f factors out as follows:

$$
\begin{array}{ccc}
\mathcal{L}_{\mathfrak{F}} & \xrightarrow{\ v\ } & [0,1] \\
e \downarrow & \equiv & \nearrow p \\
\mathcal{Q}(n) & &
\end{array}
$$

where p is the probability function in Definition 0.0.1. Hence, the semantic consequence relation $\models$ related to $\mathcal{Q}(n)$ is given by:

$$\alpha \models \beta \text{ iff } (v(\alpha), v(\beta)) \in R$$

with $R \subseteq [0,1]^2$. Since interpretations determine each possible evaluation, for each interpretation e, we denote by e_p the evaluation associated to e.
Hence, a natural extension of the classical logical consequence can be formulated as follows:

$$\alpha \models \beta \text{ iff } (e_p(\alpha) = 1 \text{ implies } e_p(\beta) = 1). \tag{1}$$

A probabilistic logic based on the consequence relation in Condition 1 was developed in [7]. Finally, let us remark that, in [6, 4], the following interesting extension of such a consequence relation was investigated:

$$\alpha \models \beta \text{ iff } e_p(\alpha) \leq e_p(\beta). \tag{2}$$

4. Probabilistic-type logic for mixed states

In general, a quantum system is not in a pure state. This may be caused, for example, by an inefficiency in the preparation procedure of the system, or else because, in practice, systems cannot be completely isolated from the environment, undergoing decoherence of their states. On the other hand, there are interesting processes that cannot be represented as unitary evolutions. A prototypical example of this phenomenon is what happens at the end of a computation process, when a non-unitary operation, a measurement, is applied, and the state becomes a probability distribution over pure states: a *mixed state*.

In view of these facts, several authors [2, 22] paid some attention to a more general model of quantum computational processes, where pure states are replaced by mixed states. This model is known as *quantum computation with mixed states*. Let us briefly describe it.

Let H be a complex Hilbert space. We denote by $\mathcal{L}(H)$ the dual space of linear operators on H. In the framework of quantum computation with mixed states, we regard a quantum state in a Hilbert space H as a *density operator* i.e., an Hermitian operator $\rho \in \mathcal{L}(H)$ that is positive semidefinite ($\rho \geq 0$) and has unit trace ($tr(\rho) = 1$). We indicate by $\mathcal{D}(H)$ the set of all density operators in H.

A *quantum operation* is a linear operator from density operators to density operators such that $\forall \rho \in \mathcal{D}(H) : \mathcal{E}(\rho) = \sum_i A_i \rho A_i^\dagger$, where A_i are operators satisfying $\sum_i A_i^\dagger A_i = I$ and $A_i^\dagger$ is the adjoint of A_i. In the representation of quantum computational processes based on mixed states, a quantum circuit is a circuit whose inputs and outputs are labeled by density operators, and whose gates are labeled by quantum operations. In terms of density operators, an n-qbit $|\psi\rangle \in \otimes^n \mathbb{C}^2$ can be represented as a matrix product $|\psi\rangle\langle\psi|$. Moreover, we can associate to any unitary operator U on a Hilbert space $\otimes^m \mathbb{C}^2$ a quantum operation $\mathcal{O}_{\mathcal{U}}$, such that, for each $\rho \in \mathcal{D}(H)$, $\mathcal{O}_{\mathcal{U}}(\rho) = \mathcal{U}\rho\mathcal{U}^\dagger$. Apparently, quantum computation with mixed states generalises the standard model based on qbits and unitary transformations. We would like to stress that the measurement process itself can be also described by a quantum operation, an important fact that strengthnes the choice of quantum operations as representatives of quantum gates. We refer to [2, 20, 22], for more details and motivations about quantum operations. In this powerful model we can naturally extend the logical basis and the notion of probability assignment defined in the qbit case. In fact, we may relate to each vector of the logical basis of $\mathbb{C}^2$ one of the distinguished density operators $P_0 = |0\rangle\langle 0|$ and $P_1 = |1\rangle\langle 1|$, that represent the falsity-property and the truth-property, respectively. The falsity and truth-properties can be generalised to any finite dimension n in the following way:

$$P_0^{(n)} = \frac{1}{tr(I^{n-1} \otimes P_0)} I^{n-1} \otimes P_0 \text{ and } P_1^{(n)} = \frac{1}{tr(I^{n-1} \otimes P_1)} I^{n-1} \otimes P_1,$$

where $n \geq 2$. By the Born rule, the probability to obtain the truth-property $P_1^{(n)}$ for a system in the state ρ is given by the following definition:

Definition 0.0.2 Let $\rho \in \mathcal{D}(\otimes^n \mathbb{C}^2)$.
Then, its *probability value* is $p(\rho) = tr(P_1^{(n)} \rho)$.

Note that, in the particular case in which $\rho = |\psi\rangle\langle\psi|$ where $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$, we obtain that $p(\rho) = |c_1|^2$. Similarly to the case of qbits, we can define a probabilistic logic based on mixed states. Consider the propositional language $\mathcal{L}_{\mathfrak{F}}$ introduced in Section 3. Here, propositional variables are interpreted as density operators in $\mathcal{D}(\otimes^n \mathbb{C}^2)$, whilst connectives are naturally interpreted as quantum operations acting on $\mathcal{D}(\otimes^n \mathbb{C}^2)$. If f is a connective in $\mathfrak{F}$, we denote by $\mathcal{E}_f$ the quantum operation associated to f in $\mathcal{L}(\mathbb{C}^2)$. An interpretation of $\mathcal{L}_{\mathfrak{F}}$ in $\mathcal{D}(\otimes^n \mathbb{C}^2)$ is any function $e : \mathcal{L}_{\mathfrak{F}} \to \mathcal{D}(\otimes^n \mathbb{C}^2)$ such that for each $f \in \mathfrak{F}$ with arity k,

$$e(f(x_1, \ldots x_k)) = \mathcal{E}_f(e(x_1) \ldots e(x_k)).$$

To define a semantic consequence relation $\models$, we also consider a natural adaptation of the notion of evaluation. Accordingly, in this setting an *evaluation* will be any function $v : \mathcal{L}_{\mathfrak{F}} \to [0, 1]$ that makes our diagram commutative:

$$\begin{array}{ccc} \mathcal{L}_{\mathfrak{F}} & \xrightarrow{v} & [0,1] \\ e \downarrow & \equiv & \nearrow p \\ \mathcal{D}(\otimes^n \mathbb{C}^2) & & \end{array}$$

where p is the probability function in Definition 0.0.2. Hence, the semantic consequence $\models$ related to $\mathcal{D}(\otimes^n \mathbb{C}^2)$ will be:

$$\alpha \models \beta \quad \text{iff} \quad (v(\alpha), v(\beta)) \in R, \tag{3}$$

with $R \subseteq [0, 1]^2$.

5. Connections with fuzzy logic

Since the Eighties, the interest in many-valued logics enormously increased. In particular, the so called *fuzzy logics*, with their truth values in the real interval $[0, 1]$, emerged as a consequence of the 1965 proposal, by L. Zadeh, of a fuzzy set theory [23]. A fundamental system of fuzzy logic, introduced by P. Hájek in [13], is known as *basic fuzzy logic*. A relevant feature of those logics is the notion of conjunction whose natural interpretation is a real valued function in $[0, 1]$, that goes under the name of continuous t-norm. More precisely, a t-norm is a continuous binary function $\odot : [0, 1]^2 \to [0, 1]$ that satisfies the following conditions:

1. $x \odot 1 = x$;
2. if $x_1 \leq x_2$ and $y_1 \leq y_2$, then $x_1 \odot y_1 \leq x_2 \odot y_2$;
3. $x \odot y = y \odot x$;
4. $x \odot (y \odot z) = (x \odot y) \odot z$.

In [16], K. Menger used the idea of t-norm in the framework of the probabilistic metric spaces. In such spaces, t-norms allow us to generalise the triangle inequality for probability distribution valued metrics. In basic fuzzy logic a genuine relationship between conjunction and implication can be established. In this system the continuity of the t-norm plays an important role. The following are the three basic continuous t-norms:

- $x \odot_P y = x \cdot y$, (Product t-norm);
- $x \odot_Ł y = \max\{x + y - 1, 0\}$, (Łukasiewicz t-norm);
- $x \odot_G y = \min\{x, y\}$, (Gödel t-norm).

These t-norms are remarkably basic, in that each possible continuous t-norm can be obtained as an *adequate combination* of them [15]. Further it is interesting to notice that these three t-norms always represent irreversible functions. In [11] is introduced a special type of quantum operations called *polynomial quantum operation*. These quantum operations can probabilistically represent any polynomial function g such that:

(1) each coefficent of g lives in $[0, 1]$;

(2) the restriction $g \upharpoonright_{[0,1]^n}$ lives in $[0, 1]$.

Further, in [11] is shown that any continuous function (not necessarily polynomial) that satisfy conditions (1) and (2) can be *approximately* represented by means of a polynomial quantum operation. Not surprisingly, the accuracy of the approximation is arbitrary (the higher is the accuracy of the approximation, the higher is the quantum operation complexity degree). The three t-norms previously introduced are continuous functions that satisfy conditions (1) and (2). Accordingly by the results mentioned above, for each of the three t-norms there exists a polynomial quantum operation that represents it. Further, in case of Product t-norm this representation is *exact* (since Product t-norm is polynomial), while in the other two cases is *approximated*.

The representation of continuous t-norms as quantum operations motivated the investigation of a logical system in the framework of probabilistic-type logic for mixed states.

Let us recall the following definition first. The *standard PMV-algebra* (*standard product multi-valued algebra*) [10, 19] is the algebra

$$[0, 1]_{PMV} = \langle [0, 1], \oplus, \odot_P, \neg, 0, 1 \rangle,$$

where $[0, 1]$ is the real unit segment, $x \oplus y = \min(1, x + y)$, the operation $\odot_P$ is the real product (corresponding to the Product t-norm introduced above), and $\neg x = 1 - x$. A slight weakening of this structure (called *quasi PMV-algebra*) plays a notable role in quantum computing, in that it describes, in a probabilistic way, a relevant quantum gates in the framework of *Poincarè irreversible quantum computational algebras* [5, 7].

As is well known, fuzzy logics (and infinite-valued Łukasiewicz logic in particular) play a relevant role in game theory and theoretical physics as shown in [17, 18], where it is investigated the deep connection between infinite-valued Łukasiewicz logic with Ulam games and $AF{-}C^*$-algebras. It would be desirable to extend this connection, by means of quasi-PMV algebras, to the investigation of quantum games and to error-correction codes in the context of quantum computation.

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