Fuzzy approach to quantum Fredkin gate

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Abstract

In the framework of quantum computation with mixed states, we introduce a fuzzy approach to the quantum Fredkin gate. Under this perspective, we investigate the behaviour of the gate applied to factorized and non-factorized quantum states.

Keywords: Fredkin gate, density operators, fuzzy logic.

1 Introduction

Standard quantum computing is based on quantum systems described by finite dimensional Hilbert spaces, specially $\mathbb{C}^2$, that is the two-dimensional space where qubits live. A qubit (the quantum counterpart of the classical bit) is represented by a unit vector in $\mathbb{C}^2$ and, generalizing for a positive integer n, n-qubits are represented by unit vectors in $\otimes^n\mathbb{C}^2$. Similar to the classical case, it is possible to study the behaviour of a number of quantum logical gates (hereafter quantum gates, for short) operating on qubits. These quantum gates are represented by unitary operators. In this work, we focus our attention in a quantum gate known as Fredkin gate. This gate is relevant since it can exactly perform all classical reversible computation. Thus, the combinational logic related to the Fredkin gate is able to represent the logic of classical reversible processes in quantum computation. Furthermore, the quantum Fredkin gate has an interesting connection with the fuzzy logic [13] which is the motivation of our article. More precisely, the aim of this article is to study a probabilistic type representation of the quantum Fredkin gate based on the Łukasiewicz negation $\neg x = 1 - x$, the Łukasiewicz sum $x \oplus y = \min x + y, 1$ and the Product t-norm $x \cdot y$ given by the usual product in the real unitary interval. Let us remark that the interval $[0,1]$ equipped with the operations $\langle \oplus, \cdot, \neg \rangle$, defines an algebraic structure called *product MV-algebra* (*PMV*-algebra for short) [6, 16] that is strictly related to the fuzzy logic.

In this way, it is possible to mathematically represent circuits made from ensemble of Fredkin gates as $\langle \oplus, \cdot, \neg \rangle$-polynomial expressions in a *PMV*-algebra. Hence, *PMV*-algebra structure related to Fredkin gates plays a similar role than Boolean algebras describing digital circuits.

From a foundation of logic viewpoint, the main result of this work is based to provide a formal analysis of the behaviour of a fuzzy type connective acting on a domain of interpretation where a kind of holistic situation is introduced. More precisely, a fuzzy representation of Fredkin gate is investigated in the special case where the gate acts on quantum systems that are generally represented by non-product states. In this way, it is possible to point out and compare the different behaviour of a logical connective that is applied to a product state (that represents a compound quantum state

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whose properties are obtained in the standard compositional way) or to a non-product state (whose properties are obtained by an holistic manner). Interestingly enough, this investigation provides a relation between fuzzy logic and holistic quantum systems.

Further, in terms of information theory, let us notice how Łukasiewicz sum and Product t-norm are also known in virtue of their relations with game theory applied to the theory of communication with feedback. Indeed, the Łukasiewicz t-norm is related to Ulam games [17] and the product t-norm is specially applied in fuzzy control theory [5] allowing to provide a probabilistic variant of Ulam game (also known as Pelc game [15]). All the arguments mentioned above also suggest possible applications of the Fredkin gate to the theory of error-correcting code in the context of quantum information.

The article is organized as follows: in Section 1 we introduce basic notions of quantum computation. Section 2 contains generalities about bipartite quantum systems. In Section 3 operational representations for Controlled and SWAP quantum gates are given. In Section 4 the quantum Fredkin gate is introduced in details. Different operational representations for this gate are provided. They turn out to be necessary to describe the fuzzy behaviour of this quantum gate. In Section 5 we study the fuzzy behaviour of the Fredkin gate acting on tripartite factorizable quantum states. In this case, the Fredkin gate can be interpreted as a ternary connective described by a $\langle \oplus, \cdot, \neg \rangle$-polynomial expression. In particular a representation of the Product t-norm is obtained. In Section 6, we study the quantum Fredkin gate acting on a bipartite quantum state. In this case, the quantum Fredkin gate is considered as a binary connective composed by a fuzzy component described by $\langle \oplus, \cdot, \neg \rangle$-polynomial expression and another component that appears in virtue of the non-separability of the quantum states. As a particular case, the gate is studied over an interesting family of non-factorizable quantum states, named *three parameters quantum states*. Finally, in Section 7 the quantum Fredkin gate is studied as a unary connective acting in the more general case of non-factorizable states obtaining a similar fuzzy representation.

2 Basic notions

In quantum computation, information is elaborated and processed by means of quantum systems. Pure states of a quantum system are described by unit vectors in a Hilbert space. A *quantum bit* or *qubit*, the fundamental concept of quantum computation, is a pure state in the Hilbert space $\mathbb{C}^2$. The *standard orthonormal basis* $\{|0\rangle, |1\rangle\}$ of $\mathbb{C}^2$ is generally called *quantum computational basis*, where $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Intuitively, $|1\rangle$ is related to the truth logical value and $|0\rangle$ to the falsity. Thus, pure states $|\psi\rangle$ in $\mathbb{C}^2$ are superpositions of the basis vectors with complex coefficients: $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$, where $|c_0|^2 + |c_1|^2 = 1$.

In the usual representation of quantum computational processes, a quantum circuit is identified by an appropriate composition of *quantum gates*, mathematically represented by *unitary operators* acting on pure states of a convenient (n-fold tensor product) Hilbert space $\otimes^n \mathbb{C}^2$ [18]. A special basis, called the 2^n-*standard orthonormal basis*, is chosen for $\otimes^n \mathbb{C}^2$. More precisely, it consists of the 2^n-orthogonal states $|\iota\rangle$, $0 \leq \iota \leq 2^n$ where ι is in binary representation and $|\iota\rangle$ can be seen as the tensor product of states i.e. the Kronecker product, $|\iota\rangle = |\iota_1\rangle \otimes |\iota_2\rangle \otimes \ldots \otimes |\iota_n\rangle$, where $\iota_j \in \{0,1\}$. It provides the standard quantum computational model, based on qubits and unitary operators.

But in general, a quantum system is not in a pure state. This may be caused, e.g., by the non-complete efficiency in the preparation procedure or by the fact that systems cannot be completely isolated from the environment, undergoing decoherence of their states. Furthermore, there are interesting processes that cannot be encoded by unitary evolutions. For example, at the end of the computation a non-unitary operation — a measurement — is applied, and the state becomes a probability distribution over pure states, or what is called a *mixed state*. In view of these facts, several authors [1, 2, 4, 9–12] have paid attention to a more general model of quantum computational processes, where pure states are replaced by mixed states that are mathematically represented by *density operators* i.e. Hermitian, positive and trace one operators. In what follows we give a short description of this mathematical model.

To each vector of the quantum computational basis of $\mathbb{C}^2$ we may associate two density operators $P_0 = |0\rangle\langle 0| = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $P_1 = |1\rangle\langle 1| = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ that represent the standard basis in this framework[1]. Let $P_1^{(n)}$ be the operator $P_1^{(n)} = (\otimes^{n-1} I) \otimes P_1$ on $\otimes^n \mathbb{C}^2$, where I is the 2×2 identity matrix. Clearly, $P_1^{(n)}$ is a 2^n-square matrix. By applying the Born rule, we consider the probability of a density operator ρ as follows:

$$p(\rho) = tr(P_1^{(n)} \rho). \tag{1}$$

We focus our attention in this probability values since it allows us to establish a link between Fredkin gate and fuzzy connectives. Note that, in the particular case in which $\rho = |\psi\rangle\langle\psi|$ where $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$, then we consider $n = 1$ obtaining $p(\rho) = |c_1|^2$. Thus, this probability value associated with ρ is the generalization, in this model, of the probability that a measurement over $|\psi\rangle$ yields $|1\rangle$ as output.

A *quantum operation* [14] is a linear operator $\mathcal{E} : \mathcal{L}(H_1) \to \mathcal{L}(H_2)$ where $\mathcal{L}(H_i)$ is the space of linear operators in the complex Hilbert space H_i $(i = 1, 2)$, representable as $\mathcal{E}(\rho) = \sum_i A_i \rho A_i^\dagger$, where A_i are operators satisfying $\sum_i A_i^\dagger A_i = I$ (Kraus representation [14]). It can be seen that a quantum operation maps density operators into density operators. Each unitary operator U on $\otimes^n \mathbb{C}^2$ gives rise to a quantum operation $\mathbb{U}$ such that $\mathbb{U}(\rho) = U\rho U^\dagger$ for any density operator ρ. We can easily see that the probability associated with the quantum operation $\mathbb{U}$ is given by:

$$p(\mathbb{U}(\rho)) = tr(P_1^{(n)}(U\rho U^\dagger)) = tr((U^\dagger P_1^{(n)} U)\rho). \tag{2}$$

The new model based on density operators and quantum operations is called '*quantum computation with mixed states*'. It allows us to represent irreversible processes as measurements in the middle of the computation.

3 Bipartite quantum states

In quantum mechanics a compound system can be represented, as a special case, by a tensor product of Hilbert spaces, each of them representing the individual parts of the system. If ρ_a and ρ_b are two density operators in the Hilbert spaces $\mathcal{H}_a$ and $\mathcal{H}_b$ respectively, the state of the compound system represented by $\rho = \rho_a \otimes \rho_b$ lives in $\mathcal{H}_a \otimes \mathcal{H}_b$ but not all density operators on $\mathcal{H}_a \otimes \mathcal{H}_b$ are expressible in this form. In particular, there are particular states lying in $\mathcal{H}_a \otimes \mathcal{H}_b$ that are not expressible as convex combination of ρ_a and ρ_b. This fact may be considered as the mathematical root of the holistic feature of quantum mechanics. In fact, there exist properties of quantum systems that characterize the whole system but that are not reducible to the local properties of its parts [3].

[1] Let us remark that, following the Dirac notation, $\langle x| = |x\rangle^\dagger$, where $\dagger$ is the conjugate transposition.

In what follows, we provide a mathematical description of quantum states based on *generalized Pauli matrices*. This framework turns out to be very useful to describe a fuzzy approach to Fredkin quantum gate. Due to the fact that the Pauli matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and I are a basis for the set of operators over $\mathbb{C}^2$, an arbitrary density operator ρ over $\mathbb{C}^2$ may be represented as

$$\rho = \frac{1}{2}(I + s_1\sigma_1 + s_2\sigma_2 + s_3\sigma_3),$$

where s_1, s_2 and s_3 are three real numbers such $s_1^2 + s_2^2 + s_3^2 \leq 1$. The triple (s_1, s_2, s_3) represents the point of the Bloch sphere that is uniquely associated with ρ. A similar canonical representation can be obtained for any n-dimensional Hilbert space by using the notion of generalized Pauli-matrices.

DEFINITION 3.1
Let $\mathcal{H}$ be a n-dimensional Hilbert space and $\{|\psi_1\rangle, \ldots, |\psi_n\rangle\}$ be the canonical othonormal basis of $\mathcal{H}$. Let k and j be two natural numbers such that: $1 \leq k < j \leq n$. Then, the *generalized Pauli-matrices* are defined as follows:

$${}^{(n)}\sigma_1^{[k,j]} = |\psi_j\rangle\langle\psi_k| + |\psi_k\rangle\langle\psi_j|$$

$${}^{(n)}\sigma_2^{[k,j]} = i(|\psi_j\rangle\langle\psi_k| - |\psi_k\rangle\langle\psi_j|)$$

and for $1 \leq k \leq n-1$

$${}^{(n)}\sigma_3^{[k]} = \sqrt{\frac{2}{k(k+1)}}(|\psi_1\rangle\langle\psi_1| + \cdots + |\psi_k\rangle\langle\psi_k| - k|\psi_{k+1}\rangle\langle\psi_{k+1}|).$$

Conveniently, let us introduce the following sets:

$${}^{(n)}\mathfrak{P}_1 = \{{}^{(n)}\sigma_1^{[k,j]} : 1 \leq k < j \leq n\}$$

$${}^{(n)}\mathfrak{P}_2 = \{{}^{(n)}\sigma_2^{[k,j]} : 1 \leq k < j \leq n\}$$

$${}^{(n)}\mathfrak{P}_3 = \{{}^{(n)}\sigma_2^{[k]} : 1 \leq k \leq n-1\}.$$

One can see that ${}^{(n)}\mathfrak{P}_1$ and ${}^{(n)}\mathfrak{P}_2$ contain $\frac{n(n-1)}{2}$ matrices, while ${}^{(n)}\mathfrak{P}_3$ contains $n-1$ matrices. Thus, if we consider the set ${}^{(n)}\mathfrak{P} = {}^{(n)}\mathfrak{P}_1 \cup {}^{(n)}\mathfrak{P}_2 \cup {}^{(n)}\mathfrak{P}_3$, it contains n^2-1 matrices. For the sake of simplicity, we consider the set ${}^{(n)}\mathfrak{P}$ ordered as follows:

$${}^{(n)}\mathfrak{P} = \{\sigma_j\}_{j=1}^{n^2-1} = \{\underbrace{\sigma_1, \ldots, \sigma_{\frac{n(n-1)}{2}}}_{{}^{(n)}\mathfrak{P}_1} \mid \underbrace{\sigma_{\frac{n(n-1)}{2}+1}, \ldots, \sigma_{n(n-1)}}_{{}^{(n)}\mathfrak{P}_2} \mid \underbrace{\sigma_{n(n-1)+1}, \ldots, \sigma_{n^2-1}}_{{}^{(n)}\mathfrak{P}_3}\}.$$

If $\mathcal{H} = \mathbb{C}^2$ one immediately obtains: ${}^{(2)}\sigma_1^{[1,2]} = \sigma_1$, ${}^{(2)}\sigma_2^{[1,2]} = \sigma_2$ and ${}^{(2)}\sigma_3^{[1]} = \sigma_3$.

Let ρ be a density operator on the n-dimensional Hilbert space $\mathcal{H}$. For any j, where $1 \leq j \leq n^2-1$, let

$$s_j(\rho) = tr(\rho\sigma_j).$$

The sequence $\langle s_1(\rho)\ldots s_{n^2-1}(\rho)\rangle$ is called the *generalized Bloch vector* associated with ρ, in view of the following well-known result [19]: let ρ be a density operator of the n-dimensional Hilbert space $\mathcal{H}$ and let σ_j be the generalized n-dimensional Pauli matrices. Then ρ can be canonically represented as follows:

$$\rho = \frac{1}{n}I^{(n)} + \frac{1}{2}\sum_{j=1}^{n^2-1} s_j(\rho)\sigma_j, \tag{3}$$

where $I^{(n)}$ is the $n \times n$ identity matrix.

A kind of converse of the above result reads: a matrix ρ having the form $\rho = \frac{1}{n}I^{(n)} + \frac{1}{2}\sum_{j=1}^{n^2-1} s_j(\rho)\sigma_j$ is a density operator if an only if its eigenvalues are non-negative. By following the Schlienz–Mahler decomposition [19], we show as any quantum bipartite state can be expressed as a sum of a factorizable state plus another quantity that represents a kind of holistic component.

Definition 3.2
Let ρ be a density operator on a n-dimensional Hilbert space $\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b$ where $dim(\mathcal{H}_a) = m$ and $dim(\mathcal{H}_b) = k$. Divide ρ in $m \times m$ blocks $B_{i,j}$, where each of them is a k-square matrix and let us consider the following matrices

$$\rho_a = tr_{\mathcal{H}_b}(\rho) = \begin{bmatrix} trB_{1,1} & trB_{1,2} & \ldots & trB_{1,m} \\ trB_{2,1} & trB_{2,2} & \ldots & trB_{2,m} \\ \vdots & \vdots & \vdots & \vdots \\ trB_{m,1} & trB_{m,2} & \ldots & trB_{m,m} \end{bmatrix}$$

$$\rho_b = tr_{\mathcal{H}_a}(\rho) = \sum_{i=1}^{m} B_{i,i}.$$

Then ρ_a is called the *partial trace (or reduced state) of* ρ *with respect to the system* $\mathcal{H}_b$ and ρ_b is called *the partial trace (or reduced state) of* ρ *with respect to the system* $\mathcal{H}_a$.

Definition 3.3
Let ρ be a density operator on a Hilbert space $\mathcal{H}_a \otimes \mathcal{H}_b$ such that $dim(\mathcal{H}_a) = m$ and $dim(\mathcal{H}_b) = k$. Then ρ is said to be (m,k)*-factorizable* iff $\rho = \rho_a \otimes \rho_b$ where ρ_m is a density operator on $\mathcal{H}_a$ and ρ_b is a density operator on $\mathcal{H}_b$.

It is well known that, if ρ is (m,k)-factorizable as $\rho = \rho_m \otimes \rho_k$, this factorization is unique.

Suppose that $\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b$ where $dim(\mathcal{H}_a) = m$ and $dim(\mathcal{H}_b) = k$. Let us consider the generalized Pauli matrices $\sigma_1^a, \ldots, \sigma_{m^2-1}^a$ and $\sigma_1^b, \ldots, \sigma_{k^2-1}^b$ arising from $\mathcal{H}_a$ and $\mathcal{H}_b$, respectively.

If we define the following coefficients:

$$M_{j,l}(\rho) = tr(\rho[\sigma_j^a \otimes \sigma_l^b]) - tr(\rho[\sigma_j^a \otimes I^{(k)}])tr(\rho[I^{(m)} \otimes \sigma_l^b])$$

and if we consider the matrix $\mathbf{M}(\rho)$ defined as

$$\mathbf{M}(\rho)=\frac{1}{4}\sum_{j=1}^{m^2-1}\sum_{l=1}^{k^2-1} M_{j,l}(\rho)(\sigma_j^a \otimes \sigma_l^b)$$

then $\mathbf{M}(\rho)$ represents the 'additional component' of ρ when ρ is not a factorized state. In this way, if ρ is a density operator on $\mathcal{H}=\mathcal{H}_a\otimes\mathcal{H}_b$, then

$$\rho=\rho_a\otimes\rho_b+\mathbf{M}(\rho). \tag{4}$$

Let us notice that $\mathbf{M}(\rho)$ is not a density operator and then it does not represent a physical state. In particular $\mathbf{M}(\rho)$ is an hermitian operator in $\mathcal{H}_a\otimes\mathcal{H}_b$. We refer to $\mathbf{M}(\rho)$ as the *Hermitian component* of ρ.

4 Reversible computation: controlled and SWAP quantum gates

In classical computation several models are investigated. Boolean circuit is by far the easiest model to extend to get a quantum computational model. Boolean circuits are built by ensembles of digital gates *NOT*, *AND*, *OR*, etc. and they are mathematically represented by Boolean functions of the form $f:\{0,1\}^n\to\{0,1\}^m$. Digital circuits (as the simple gates *AND* and *OR*) are generally not reversible, in the sense that the Boolean function that they represent is not invertible.

In devising a model of a quantum computer, we generalize the notion of digital circuit for classical computation. But the logic related to quantum gates must be ruled by unitary evolutions; hence, the gates generally represent invertible transformations. It is clear that any irreversible computation can be represented as an evaluation of an invertible function. For example, for any function $f:\{0,1\}^n\to\{0,1\}^m$ it is possible to built a function $f^*:\{0,1\}^{n+m}\to\{0,1\}^{n+m}$ such that $f^*(x,0^{(m)})=(x,f(x))$ where $0^{(m)}$ denotes a set of m zero bits. From the classical computational standpoint, f^* provides redundant information and we can discard it without affecting the outcome of the computation. However, in quantum computing, discarding information can drastically change the outcome of a computation. In this way, quantum circuits that extend the classical circuits are based on reversible classical computation. In this section, we extend two classes of reversible classical gates: the Controlled and SWAP gates that are useful to provide the quantum version of the Fredkin gate, as will be introduced in the next section.

Let $U:\{0,1\}^n\to\{0,1\}^n$ be a reversible classical gate. The *Controlled-U* gate is defined as $CU^{(n)}:\{0,1\}^{n+1}\to\{0,1\}^{n+1}$ such that

$$CU^{(n)}(x_1,\cdots,x_{n+1})=\begin{cases}(x_1,\cdots,x_{n+1}) & \text{if } x_1=0;\\ (x_1,U^{(n)}(x_2,\cdots,x_{n+1})) & \text{if } x_1=1.\end{cases}$$

The natural quantum extension of a classical Controlled gate is carried out in the following way:

DEFINITION 4.1
Let $U^{(n)}$ be an arbitrary unitary operator representing a quantum gate on $\otimes^n\mathbb{C}^2$. A Controlled-U gate $CU^{(n)}$ is a linear operator on $\otimes^{(n+1)}\mathbb{C}^2$ such that for any vector $|x_1,\cdots,x_{n+1}\rangle$ in the standard computational basis of $\otimes^{n+1}\mathbb{C}^2$, is:

$$CU^{(n)}|x_1,\cdots,x_{n+1}\rangle=\begin{cases}|x_1,\cdots,x_{n+1}\rangle & \text{if } x_1=0;\\ |x_1\rangle\otimes U^{(n)}|x_2,\cdots,x_{n+1}\rangle & \text{if } x_1=1.\end{cases}$$

Simply speaking, $CU^{(n)}$ gate

- leaves the first (control) qubit $|x_1\rangle$ inalterate;
- acts on the other n qubits $|x_2,\cdots,x_{n+1}\rangle$ as the identity $I^{(n)}$ if $x_1=0$ and as the $U^{(n)}$ operator if $x_1=1$.

By following the standard construction of controlled operators (see, *e.g.,* section 4.3 in [18]), if $U^{(n)}$ is a unitary n-qubit gate, then the Controlled-U is a linear operator on $\otimes^{n+1}\mathbb{C}^2$ that assumes the following representation:

$$CU^{(n)}=\left[\begin{array}{c|c} I^{(n)} & 0 \\ \hline 0 & U^{(n)} \end{array}\right] = \left[\begin{array}{c|c} I^{(n)} & 0 \\ \hline 0 & 0 \end{array}\right]+\left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & U^{(n)} \end{array}\right] = P_0\otimes I^{(n)}+P_1\otimes U^{(n)}.$$

This linear operator admits a natural generalization to the case of n-Control unitary $CU^{(n,m)}$ operators, that are represented by:

$$\begin{aligned} CU^{(n,m)} &= I^{(n-1)}\otimes\left[\begin{array}{c|c} I^{(m)} & 0 \\ \hline 0 & U^{(m)} \end{array}\right]= \\ &= P_0^{(n)}\otimes I^{(m)}+ P_1^{(n)}\otimes U^{(m)} = I^{(n-1)}\otimes\left(P_0\otimes I^{(m)}+P_1\otimes U^{(m)}\right). \end{aligned} \tag{5}$$

The classical SWAP gate is given by the function $SWAP:\{0,1\}^2\rightarrow\{0,1\}^2$ such that $SWAP(x,y)=(y,x)$. A useful generalization of this gate is $SWAP^{(m,l)}:\{0,1\}^{m+l}\rightarrow\{0,1\}^{m+l}$ such that

$$SWAP^{(m,l)}(y_1,\ldots,y_m,z_1,\ldots,z_l)=(y_1,\ldots,y_{m-1},z_l,z_1,\ldots,z_{l-1},y_m).$$

A suitable extension of the $SWAP^{(m,l)}$ into the context of quantum logical gate is the following.

DEFINITION 4.2
Let $|x\rangle=|x_1,x_2,\ldots,x_m\rangle$ and $|y\rangle=|y_1,y_2,\ldots,y_l\rangle$ be vectors of the standard orthonormal basis in $\otimes^m\mathbb{C}^2$ and $\otimes^l\mathbb{C}^2$, respectively. Then, the quantum $\mathrm{SWAP}^{(m,l)}$ gate is defined as:

$$\mathrm{SWAP}^{(m,l)}|y_1,\ldots,y_m\rangle|z_1,\ldots,z_l\rangle=|y_1,\ldots,y_{m-1},z_l\rangle|z_1,\ldots,z_{l-1},y_m\rangle.$$

The following proposition describes the matrix representation of $SWAP^{(m,l)}$.

PROPOSITION 4.1
[20] The matrix form of $SWAP^{(m,l)}$ is provided by the following unitary matrix:

$$\begin{aligned} SWAP^{(m,l)} &= I^{(m-1)}\otimes\left[\begin{array}{c|c} P_0^{(l)} & L_1^{(l)} \\ \hline L_0^{(l)} & P_1^{(l)} \end{array}\right]= \\ &= P_0^{(m)}\otimes P_0^{(l)}+P_1^{(m)}\otimes P_1^{(l)}+L_0^{(m)}\otimes L_1^{(l)}+L_1^{(m)}\otimes L_0^{(l)}, \end{aligned}$$

where $L_0 = |0\rangle\langle 1|$ and $L_1 = |1\rangle\langle 0|$ are the so called Ladder operators and $L_0^{(l)} = I^{(l-1)} \otimes L_0$ and $L_1^{(l)} = I^{(l-1)} \otimes L_1$.

5 Fredkin gate

The Fredkin gate [7] is a *reversible logic gate* of the form $\{0,1\}^3 \to \{0,1\}^3$ that is universal for classical computation, i.e., any digital reversible circuit can be built from an ensemble of Fredkin gates alone. Formaly, this gate is defined as follows:

DEFINITION 5.1
The application of the Fredkin gate to a sequence of three bits (x,y,z) is dictated by:

$$F(x,y,z) = (x,\ y \widehat{+} x(y \widehat{+} z),\ z \widehat{+} x(y \widehat{+} z)),$$

where, $\widehat{+}$ is the sum modulo 2.

Briefly, the behaviour of the Fredkin gate can be described in the following way: the first bit x is a control bit, that remains unaffected by the action of the gate. The second and the third bits y and z are the target bits that are swapped if and only if the control bit x is 1, and kept unchanged otherwise. In other words, the Fredkin gate can be understood as a *Control-SWAP gate*.

The Fredkin gate can be used to represent the classical NOT, AND and OR gates in the following way:

- F(x,0,1)=(x, - ,NOT(x));
- F(x,y,0)=(x, - ,AND(x,y));
- F(x,y,1)=(x, OR(x,y), -),

showing the universality of this gate for the classical logic.

The Fredkin gates can also be naturally extended as a quantum gate in the following way.

DEFINITION 5.2
Let $|x\rangle = |x_1, x_2, \ldots, x_n\rangle$, $|y\rangle = |y_1, y_2, \ldots, y_m\rangle$ and $|z\rangle = |z_1, z_2 \ldots, z_l\rangle$ be vectors of the standard orthonormal basis in $\otimes^n \mathbb{C}^2$, $\otimes^m \mathbb{C}^2$ and $\otimes^l \mathbb{C}^2$, respectively. Then, the quantum Fredkin gate is defined by the following equation:

$$F^{(n,m,l)}|x,y,z\rangle = |x\rangle|y_1 \ldots y_{m-1},\ y_m \widehat{+} x_n(y_m \widehat{+} z_l)\rangle|z_1 \ldots z_{l-1},\ z_l \widehat{+} x_n(y_m \widehat{+} z_l)\rangle.$$

Notice that $F^{(n,m,l)}$ is a linear operator in $\otimes^{(n+m+l)}\mathbb{C}^2$.

To obtain a matrix representation for $F^{(n,m,l)}$, we take into account that the classical Fredkin gate behaves as a *Control-SWAP* gate. Hence, in virtue of the Equation (5), we end up with the following representation of the quantum Fredkin gate.

PROPOSITION 5.1
For any natural number $n,m,l \geq 1$, the quantum Fredkin gate $F^{(n,m,l)}$ assumes the following matrix representation:

$$F^{(n,m,l)} = P_0^{(n)} \otimes I^{(m+l)} + P_1^{(n)} \otimes SWAP^{(m,l)} =$$
$$= \begin{bmatrix} I^{(m+l)} & \dots & & & \\ \vdots & Swap^{(m,l)} & & & \\ & & I^{(m+l)} & & \\ & & & \ddots \; 2^n - times & \\ & & & & Swap^{(m,l)} \end{bmatrix}.$$

■

Now, let us recall that, by Definition 5.2 and as described in the previous section, also the quantum Fredkin gate can be considered as a *Control-SWAP gate*. Therefore:

$$F^{(n,m,l)} = I^{(n-1)} \otimes \left[\begin{array}{c|c} I^{(m+l)} & 0 \\ \hline 0 & SWAP^{(m,l)} \end{array} \right] \tag{6}$$
$$= I^{(n-1)} \otimes \left(P_0 \otimes I^{(m+l)} + P_1 \otimes SWAP^{(m,l)} \right). \tag{7}$$

Further, recalling that from the block representation of $SWAP^{(m,l)}$ easily turns out the unitarity of the *SWAP* gate, by very similar considerations also the unitarity of $F^{(n,m,l)}$ can be proven.

The operational and the matrix representations of the quantum Fredkin gate (from now on Fredkin gate, for short) turn out to be very useful to achieve easy computations of several algebraic properties of this gate. In the next section, we provide a calculation to evaluate how the probability of a factorized density operator changes after the application of the Fredkin gate.

6 Fuzziness in the Fredkin gate

In this section, we establish a relation between the Fredkin gate and the fuzzy logic of continuous t-norm based on $\langle \oplus, \cdot, - \rangle$- polynomial expression.

PROPOSITION 6.1
Let us consider the factorized density operator $\rho = \rho_n \otimes \rho_m \otimes \rho_l$ where ρ_i acts on $\otimes^i \mathbb{C}^2$ for $i \in \{n,m,l\}$. Then,

$$p(\mathbb{F}^{(n,m,l)}(\rho)) = \neg p(\rho_n) \cdot p(\rho_l) \oplus p(\rho_n) \cdot p(\rho_m).$$

PROOF. By Equation (2), we have that:

$$p(\mathbb{F}^{(n,m,l)}(\rho)) = Tr(P_1^{(n+m+l)} \cdot \left(F^{(n,m,l)} \cdot \rho \cdot F^{(n,m,l)} \right))$$
$$= Tr(\left(F^{(n,m,l)} \cdot P_1^{(n+m+l)} \cdot F^{(n,m,l)} \right) \rho).$$

By using the representation of $F^{(n,m,l)}$ provided in Proposition 5.1, we obtain:

$$F^{(n,m,l)} \cdot P_1^{(n+m+l)} \cdot F^{(n,m,l)} =$$
$$= I^{(n-1)} \otimes \Big[\Big(P_0 \otimes I^{(m+l)} + P_1 \otimes SWAP^{(m,l)} \Big) \cdot \Big(I^{(m+l)} \otimes P_1 \Big) \cdot \Big(P_0 \otimes I^{(m+l)} + P_1 \otimes SWAP^{(m,l)} \Big) \Big] =$$
$$= I^{(n-1)} \otimes [(P_0 \cdot I \cdot P_0) \otimes (I^{(m+l)} \cdot P_1^{(m+l)} \cdot I^{(m+l)}) +$$
$$+ (P_0 \cdot I \cdot P_1) \otimes (I^{(m+l)} \cdot (I^{(m+l-1)} \otimes P_1) \cdot SWAP^{(m,l)}) +$$
$$+ (P_1 \cdot I \cdot P_0) \otimes (SWAP^{(m,l)} \cdot (I^{(m+l-1)} \otimes P_1) \cdot I^{(m+l)})$$
$$+ (P_1 \cdot I \cdot P_1) \otimes (SWAP^{(m,l)} \cdot P_1^{(m+l)} \cdot SWAP^{(m,l)})].$$

Let us recall that:

- $P_0 \cdot I \cdot P_1 = P_1 \cdot I \cdot P_0 = 0$;
- $SWAP^{(m,l)} = I^{(m-1)} \otimes SWAP^{(1,l)}$.

Further, let us notice that

$$(P_0 \cdot I \cdot P_0) \otimes (I^{(m+l)} \cdot P_1^{(m+l)} \cdot I^{(m+l)}) = P_0 \otimes P_1^{(m+l)}$$
$$\text{and } (P_1 \cdot I \cdot P_1) \otimes (SWAP^{(m,l)} \cdot P_1^{(m+l)} \cdot SWAP^{(m,l)}) =$$
$$= P_1 \otimes ((I^{(m-1)} \otimes SWAP^{(1,l)}) \cdot (I^{(m-1)} \otimes P_1^{(l+1)}) \cdot (I^{(m-1)} \otimes SWAP^{(1,l)})) =$$
$$= P_1 \otimes (I^{(m-1)} \cdot I^{(m-1)} \cdot I^{(m-1)}) \otimes (SWAP^{(1,l)} \cdot (I \otimes P_1^{(l)}) \cdot SWAP^{(1,l)}) =$$

(by Definition of *SWAP*) $= P_1 \otimes I^{(m-1)} \otimes (P_1 \otimes I^{(l)})$. Hence,
$F^{(n,m,l)} \cdot P_1^{(n+m+l)} \cdot F^{(n,m,l)} =$

$$= I^{n-1} \otimes (P_0 \otimes P_1^{(m+l)} + P_1 \otimes I^{(m-1)} \otimes P_1 \otimes I^{(l)})) =$$
$$= P_0^{(n)} \otimes I^{(m)} \otimes P_1^{(l)} + P_1^{(n)} \otimes P_1^{(m)} \otimes I^{(l)} =$$
$$= (I^{(n)} - P_1^{(n)}) \otimes I^{(m)} \otimes P_1^{(l)} + P_1^{(n)} \otimes P_1^{(m)} \otimes I^{(l)}.$$

Therefore, the probability of the Fredkin operation on a product state $\rho = \rho_n \otimes \rho_m \otimes \rho_l$ is given by:

$$p(\mathbb{F}^{(n,m,l)}(\rho)) = Tr\Big[\Big(\Big(I^{(n)} - P_1^{(n)} \Big) \otimes I^{(m)} \otimes P_1^{(l)} \; + \; P_1^{(n)} \otimes P_1^{(m)} \otimes I^{(l)} \Big) \cdot (\rho_n \otimes \rho_m \otimes \rho_l) \Big]$$

which can be reduced in a straightforward manner to $(1 - p(\rho_n))\, p(\rho_l) \; + \; p(\rho_n) p(\rho_m)$.

Let us notice that, since $0 \le p(\mathbb{F}^{(n,m,l)}(\rho_n \otimes \rho_m \otimes \rho_l)) \le 1$, the above sum is a Łukasiewicz sum that can be written as $\neg p(\rho_n) \cdot p(\rho_l) \, \oplus \, p(\rho_n) \cdot p(\rho_m)$. ■

In the special case where the third component $\rho_l = P_0$, then the probability of the Fredkin gate behaves as the product *t*-norm:

$$p(\mathbb{F}^{(n,m,1)}(\rho_n \otimes \rho_m \otimes P_0)) = p(\rho_n) \cdot p(\rho_m). \tag{8}$$

Let us notice that the probability values of the Fredkin gate obtained in Proposition 6.1 and in Equation (8) are polynomial terms interpreted in the standard product MV-algebra defined in the real interval $[0,1]$. Thus, it describes a fuzzy behaviour of the Fredkin gate as a polynomial term in the language of PMV-algebras.

7 Fredkin gate as binary fuzzy connective

Up to now, we have studied the Fredkin gate as a ternary connective, inheriting this condition from classical logic. However, if the non-separability condition of the input state is considered, then, from a logical point of view, the arity of the Fredkin gate could change.

Real quantum systems continually interact with the environment, building up correlations. For a more realistic approach, we can assume that the input of an arbitrary quantum gate can also be a non-factorizable state on $\otimes^{n+m+1}\mathbb{C}^2$.

To study the fuzziness of the Fredking gate in this general case, in this section we consider the particular case where the input is of the form $\rho\otimes\sigma$ where ρ, that acts on $\otimes^{(n+m)}\mathbb{C}^2$, is a non-factorizable state and σ acts on $\mathbb{C}^2$. In this way, the Fredkin gate $F^{(n,m,1)}$ that operates on the space $\otimes^{(n+m+1)}\mathbb{C}^2$ can be seen as a binary logical connective of the form $F^{(n,m,1)}(\rho\otimes\sigma)$. Further, the results obtained in this section will be used in the next section to study the fuzziness of the Fredkin gate on a general non-factorizable state ρ on $\otimes^{(n+m+1)}\mathbb{C}^2$.

Let ρ be a density operator acting on $\otimes^{n+m}\mathbb{C}^2$ and let us denote by r_i the i-th diagonal element of ρ, with $1\leq i\leq 2^{n+m}$. In this way we consider a partition of the diagonal of ρ in 2^n blocks, each one containing 2^m elements:

$$diag(\rho)=[(r_1,\cdots,r_{2^m}),(r_{2^m+1},\cdots,r_{2^{m+1}}),\cdots,(r_{(2^n-1)2^m+1},\cdots,r_{2^{m+n}})].$$

For the sake of the simplicity, we can indicate by B_i (with $1\leq i\leq 2^n$) the i-th block containing 2^m elements of $diag(\rho)$. Hence,

$$diag(\rho)=[B_1,B_2,\cdots,B_{2^n}].$$

Moreover, we introduce the following parameters:

$\alpha^{n,m}(\rho)=\sum_{i=1}^{2^{n-1}}\sum_{j=1}^{2^{m-1}} r_{(2i-1)2^m+2j}$ i.e. the sum of the even diagonal elements of the even blocks of $diag(\rho)$,
$\beta^{n,m}(\rho)=\sum_{i=1}^{2^{n-1}}\sum_{j=1}^{2^{m-1}} r_{(2i-1)2^m+2(j-1)}$ i.e. the sum of the odd diagonal elements of the even blocks of $diag(\rho)$,
$\gamma^{n,m}(\rho)=\sum_{i=1}^{2^{n-1}}\sum_{j=1}^{2^{m-1}} r_{(2i-2)2^m+2j}$ the sum of the even diagonal elements of the odd blocks of $diag(\rho)$,
$\delta^{n,m}(\rho)=\sum_{i=1}^{2^{n-1}}\sum_{j=1}^{2^{m-1}} r_{(2i-2)2^m+2(j-1)}$ the sum of the odd diagonal elements of the odd blocks of $diag(\rho)$.

PROPOSITION 7.1
Let ρ be a density operator on $\otimes^{n+m}\mathbb{C}^2$. We have that:

$$p(\mathbb{F}^{(n,m,1)}(\rho\otimes P_0))=\alpha^{n,m}(\rho).$$

PROOF. First, let us consider the matrix form of

$$\rho\otimes P_0=\begin{bmatrix} r_1\otimes P_0 & \dots & & \\ \vdots & r_2\otimes P_0 & & \\ & & \ddots & \\ & & & r_{2^{n+m}}\otimes P_0 \end{bmatrix}.$$

Hence, the diagonal entries of $\rho \otimes P_0$ are given by

$$diag(\rho \otimes P_0) = r_1, 0, r_2, 0, \cdots, r_{2^{n+m}-1}, 0, r_{2^{n+m}}.$$

The $diag(\rho \otimes P_0)$ can be partitioned into 2^n blocks with lenght 2^{m+1}, as follows:

$$\begin{aligned} diag(\rho \otimes P_0) &= [r_1, 0, r_2, 0, \cdots, r_{2^m}, 0], [r_{2^m+1}, 0, r_{2^m+2}, 0, \cdots, r_{2^{m+1}}, 0], \cdots, \\ &\quad \cdots, [r_{(2^n-1)2^m+1}, 0, \cdots, r_{2^{m+n}}, 0]. \end{aligned}$$

$$\begin{aligned} F^{(n,m,1)} &= P_0^{(n)} \otimes I^{(m+1)} + P_1^{(n)} \otimes Swap^{(m,1)} \\ &= \begin{bmatrix} I^{(m+1)} & \cdots & & & \\ \vdots & Swap^{(m,1)} & & & \\ & & I^{(m+1)} & & \\ & & & \ddots & \\ & & & & Swap^{(m,1)} \end{bmatrix}. \end{aligned}$$

Hence, the application of $\mathbb{F}^{(n,m,1)}$ to $\rho \otimes P_0$, corresponds to apply the identity $I^{(m+1)}$ to the odd diagonal blocks of $\rho \otimes P_0$ and the $Swap^{(m,1)}$ to the even diagonal blocks of $\rho \otimes P_0$, respectively.

Let us consider a density operator σ on $\otimes^{m+1}\mathbb{C}^2$. Let $diag(\sigma) = s_1, s_2, \cdots, s_{2^{m+1}}$. It is possible to divide $diag(\sigma)$ into 2^{m-1} blocks with length 4 as follows: $diag(\sigma) = [s_1, s_2, s_3, s_4], \cdots, [s_{2^{m+1}-3}, s_{2^{m+1}-2}, s_{2^{m+1}-1}, s_{2^{m+1}}]$. It is easy to see that

$$diag(Swap^{(m,1)} \sigma Swap^{(m,1)}) = [s_1, s_3, s_2, s_4], \cdots, [s_{2^{m+1}-3}, s_{2^{m+1}-1}, s_{2^{m+1}-2}, s^{2^{m+1}}].$$

In other words, the extremes of each block are left unchanged while the intermediate elements are swapped. Therefore, we have that:

$$\begin{aligned} & diag(\mathbb{F}^{n,m,1}(\rho \otimes P_0)) = \\ = & [r_1, 0, r_2, 0, \cdots, r_{2^m}, 0], [r_{2^m+1}, r_{2^m+2}, 0, 0, r_{2^m+3}, r_{2^m+4}, 0, 0, \cdots, 0, 0, r_{2^{m+1}-1}, r_{2^{m+1}}], \cdots, \\ & [r_{(2^n-1)2^m+1}, r_{(2^n-1)2^m+2}, 0, 0, \cdots, 0, 0, r_{2^{m+n}-1}, r_{2^{m+n}}]. \end{aligned}$$

Finally, $p(\mathbb{F}^{(n,m,1)}(\rho \otimes P_0)) = Tr(P_1^{(n+m+1)} \mathbb{F}^{(n,m,1)}(\rho \otimes P_0))$ that corresponds to select only the even entries of $diag(\mathbb{F}^{n,m,1}(\rho \otimes P_0))$. But all the even entries of the odd blocks of $diag(\mathbb{F}^{n,m,1}(\rho \otimes P_0))$ are null and the even entries of the even blocks of $diag(\mathbb{F}^{n,m,1}(\rho \otimes P_0))$ correspond to the even entries of the even blocks of ρ. Hence, our claim. ■

Now we consider the more general case where the input of the Fredkin operation is given by $\rho \otimes \sigma$, where ρ acts on $\otimes^{n+m}\mathbb{C}^2$ and σ acts on $\mathbb{C}^2$.

THEOREM 7.1
Let ρ be a density operator on $\otimes^{n+m}\mathbb{C}^2$ and σ be a density operator on $\mathbb{C}^2$ two density operators. We have that:

$$p(\mathbb{F}^{(n,m,1)}(\rho \otimes \sigma)) = (1 - p(\sigma))\alpha^{n,m}(\rho) + p(\sigma)(1 - \beta^{n,m}(\rho)).$$

PROOF. An arbitrary density operator σ on $\mathbb{C}^2$ can be written as:

$$\sigma = (1-s)P_0 + sP_1 + aL_0 + a^*L_1,$$

where $s = p(\sigma)$, $a \in \mathbb{C}$, L_0 and L_1 are the Ladder operators defined above.

$$\begin{aligned} p(\mathbb{F}^{(n,m,1)}(\rho\otimes\sigma)) &= TrP_1^{(n+m+1)}(\mathbb{F}(\rho\otimes((1-s)P_0+sP_1+aL_0+a^*L_1))) = \\ &= (1-s)TrP_1^{(n+m+1)}\mathbb{F}(\rho\otimes P_0) + sTrP_1^{(n+m+1)}\mathbb{F}(\rho\otimes P_1) + \\ &+ aTrP_1^{(n+m+1)}\mathbb{F}(\rho\otimes L_0) + a^*TrP_1^{(n+m+1)}\mathbb{F}(\rho\otimes L_1). \end{aligned}$$

It is easy to verify that $TrP_1^{(n+m+1)}\mathbb{F}(\rho\otimes L_0) = TrP_1^{(n+m+1)}\mathbb{F}(\rho\otimes L_1) = 0$.
Hence:

$$\begin{aligned} p(\mathbb{F}^{(n,m,1)}(\rho\otimes\sigma)) &= (1-s)TrP_1^{(n+m+1)}\mathbb{F}(\rho\otimes P_0) + sTrP_1^{(n+m+1)}\mathbb{F}(\rho\otimes(I-P_0)) = \\ &= (1-2s)TrP_1^{(n+m+1)}\mathbb{F}(\rho\otimes P_0) + sTrP_1^{(n+m+1)}\mathbb{F}(\rho\otimes I). \end{aligned}$$

By Proposition 7.1, we already have that $TrP_1^{(n+m+1)}\mathbb{F}(\rho\otimes P_0) = \alpha^{n,m}$.
Moreover, we have that

$$\begin{aligned} diag(\rho\otimes I) &= [r_1, r_1, r_2, r_2, \cdots, r_{2^m}, r_{2^m}], \\ &\quad [r_{2^m+1}, r_{2^m+1}, r_{2^m+2}, r_{2^m+2}, \cdots, r_{2^{m+1}}, r_{2^{m+1}}], \cdots \\ &\cdots, [r_{(2^n-1)2^m+1}, r_{(2^n-1)2^m+1}, r_{(2^n-1)2^m+2}, r_{(2^n-1)2^m+2}, \cdots, r_{2^{m+n}}, r_{2^{m+n}}]. \end{aligned}$$

Hence,

$$\begin{aligned} diag(\mathbb{F}^{n,m,1}(\rho\otimes I)) &= [r_1, r_1, r_2, r_2, \cdots, r_{2^m}, r_{2^m}], \\ &\quad [r_{2^m+1}, r_{2^m+2}, r_{2^m+1}, r_{2^m+2}, \cdots, r_{2^{m+1}-1}, r_{2^{m+1}}], \cdots \\ &\cdots, [r_{(2^n-1)2^m+1}, r_{(2^n-1)2^m+2}, r_{(2^n-1)2^m+1}, r_{(2^n-1)2^m+2}, \cdots, r_{2^{m+n}-1}, r_{2^{m+n}}]. \end{aligned}$$

Let us remind that $TrP_1^{n+m+1}(\mathbb{F}^{(n,m,1)}(\rho\otimes I))$ sums only the even elements of $(\mathbb{F}^{(n,m,1)}(\rho\otimes I))$, then we have that $TrP_1^{n+m+1}(\mathbb{F}^{(n,m,1)}(\rho\otimes I))$ is equal to the sum of all the entries of the odd blocks of ρ with the double of the even entries of the even blocks of ρ. Hence, $p(\mathbb{F}^{(n,m,1)}(\rho\otimes I)) = \gamma^{n,m}(\rho) + \delta^{n,m}(\rho) + 2\alpha^{n,m}(\rho) = 1 - \beta^{n,m}(\rho) + \alpha^{n,m}(\rho)$, that allows us to conclude that:

$$\begin{aligned} p(\mathbb{F}^{(n,m,1)}(\rho\otimes\sigma)) &= (1-2s)TrP_1^{(n+m+1)}\mathbb{F}(\rho\otimes P_0) + sTrP_1^{(n+m+1)}\mathbb{F}(\rho\otimes I) = \\ &= (1-2s)\alpha^{n,m}(\rho) + s(1-\beta^{n,m}(\rho)+\alpha^{n,m}(\rho)) = \\ &= (1-p(\sigma))\alpha^{n,m}(\rho) + p(\sigma)(1-\beta^{n,m}(\rho)). \end{aligned}$$

■

By Definition 4 it is straightforward to see the following result.

LEMMA 7.1
Let ρ be a density operator on $\otimes^{n+m}\mathbb{C}^2$ and let $\rho_n = tr_{\otimes^m\mathbb{C}^2}(\rho)$ and $\rho_m = tr_{\otimes^n\mathbb{C}^2}(\rho)$ the reduced states of ρ.

(1) $p(\rho_n) = \alpha^{n,m}(\rho) + \beta^{n,m}(\rho)$;
(2) $p(\rho_m) = \alpha^{n,m}(\rho) + \gamma^{n,m}(\rho) = p(\rho)$.

■

By Equation (4), we have seen that any ρ on $\otimes^{n+m}\mathbb{C}^2$ can be decomposed as $\rho = \rho_n \otimes \rho_m + M(\rho)$, where ρ_n and ρ_m are the reduced states of ρ with respect to the subsystems $\otimes^m\mathbb{C}^2$ and $\otimes^n\mathbb{C}^2$ respectively and $M(\rho)$ is an 'additional component' whose trace is null. Therefore, $M(\rho)$ has not any influence on the value of $p(\rho)$. However, if we consider a density operator σ on $\mathbb{C}^2$, then the incidence of $M(\rho)$ on the value of $p(\mathbb{F}^{(n,m,1)}(\rho\otimes\sigma))$ is no longer negligible. To point out the fuzziness of the Fredkin gate with respect to $\rho\otimes\sigma$ acting on $\otimes^{n+m+1}$ we end up with the following result.

THEOREM 7.2
For any density operators ρ on $\otimes^{n+m}\mathbb{C}^2$ and σ on $\mathbb{C}^2$, we have that:

$$\begin{aligned} p(\mathbb{F}^{(n,m,1)}(\rho\otimes\sigma)) &= (\neg p(\rho_n)\cdot p(\sigma) \oplus p(\rho_n)\cdot p(\rho_m)) + \\ &\quad + \alpha^{n,m}(\rho)\cdot(1+p(\rho)) + \beta^{n,m}(\rho)\cdot p(\rho), \end{aligned}$$

where ρ_n and ρ_m are the reduced states of ρ with respect to the subsystems $\otimes^m\mathbb{C}^2$ and $\otimes^n\mathbb{C}^2$, respectively.

PROOF. By Equation (4), we have that $\rho \in \otimes^{n+m}\mathbb{C}^2$ can be decomposed as $\rho = \rho_n \otimes \rho_m + M(\rho)$. Then, by the linearity of the trace we have:

$$\begin{aligned} p(\mathbb{F}^{(n,m,1)}(\rho\otimes\sigma)) &= p(\mathbb{F}^{(n,m,1)}((\rho_n\otimes\rho_m + M(\rho))\otimes\sigma)) = \\ &= p(\mathbb{F}^{(n,m,1)}(\rho_n\otimes\rho_m\otimes\sigma)) + p(\mathbb{F}(M(\rho)\otimes\sigma))) \end{aligned}$$

Thus,

$$p(\mathbb{F}^{(n,m,1)}(M(\rho)\otimes\sigma)) = p(\mathbb{F}^{(n,m,1)}(\rho\otimes\sigma)) - p(\mathbb{F}^{(n,m,1)}(\rho_m\otimes\rho_n\otimes\sigma)).$$

By Theorem 7.1, Lemma 7.1 and Proposition 6.1, we have that:

$$p(\mathbb{F}^{(n,m,1)}(M(\rho)\otimes\sigma)) = \alpha^{n,m}(\rho) - p(\rho)(\alpha^{n,m}(\rho) + \beta^{n,m}(\rho)).$$

Hence, our claim. ■

As a remark, let us notice that the fuzziness of the Fredkin gate applied to $\rho\otimes\sigma$ (with ρ generally non-factorizable) appears in terms of the two reduced states ρ_n and ρ_m of ρ. Further, interestingly enough, we note that the quantity $p(\mathbb{F}^{(n,m,1)}(M(\rho)\otimes\sigma))$ does not explicitly depend on σ. In other words, the quantity $\beta^{n,m}(\rho)p(\rho)$ that is an additional term with respect to the 'fuzzy component' $\alpha^{n,m}(\rho)(1+p(\rho))$, only depends on the non-factorizability of the state ρ.

In the following example we show the behaviour of the Fredkin gate applied to a family of non-factorized states.

EXAMPLE 7.1
Let us consider the following family of quantum states $\rho_{abc}\in\otimes^2\mathbb{C}^2$ (called *three parameters states [8]*):

$$\rho_{abc}=\frac{1}{4}\begin{bmatrix}1+a & 0 & 0 & 0\\ 0 & 1-b & ic & 0\\ 0 & -ic & 1+b & 0\\ 0 & 0 & 0 & 1-a\end{bmatrix},$$

where a,b,c are real parameters such that $a^2\leq 1$ and $b^2+c^2\leq 1$. It can be proved that ρ_{abc} represents a separable state if and only if $a^2+c^2\leq 1$.

By a straightforward calculation and by Theorem 7.2, it can be seen that:

(1) $p(\mathbb{F}^{(1,1,1)}(\rho_{abc}\otimes P_0))=\frac{1-a}{4}$,
(2) $p(\rho_{abc_1})p(\rho_{abc_2})=\frac{(a-2)^2-b^2}{16}$ (where ρ_{abc_i} is the reduced state of ρ_{abc_i} with respect to the i-th component),
(3) $p(\mathbb{F}^{1,1,1}M(\rho_{abc}\otimes P_0))=\frac{b^2-a^2}{4}$.

In Figure 1, the upper surface represents the fuzzy component of $p(\mathbb{F}^{(1,1,1)}(\rho_{abc}\otimes P_0))$ (given in the item 2)) and the lower surface represents the quantity $p(\mathbb{F}^{1,1,1}M(\rho_{abc}\otimes P_0))$ given in the third item. While, in Figure 2 we represent the probability value of $\mathbb{F}^{1,1,1}(\rho_{abc}\otimes P_0)$.

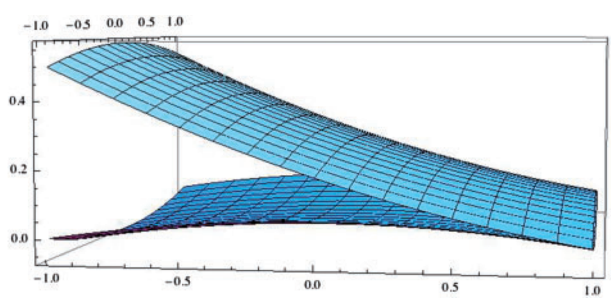

FIGURE 1. The upper surface represents the fuzzy component of $p(\mathbb{F}^{(1,1,1)}(\rho_{abc}\otimes P_0))$; the lower surface represents the quantity $p(\mathbb{F}^{1,1,1}M(\rho_{abc}\otimes P_0))$.

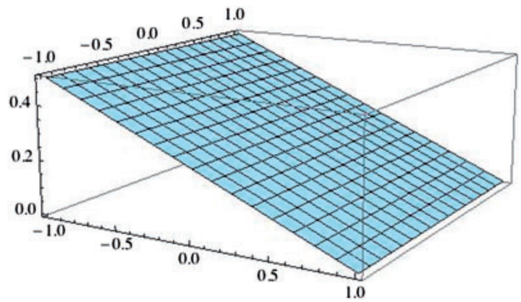

FIGURE 2. The probability value of $\mathbb{F}^{1,1,1}(\rho_{abc} \otimes P_0)$.

8 Fredkin gate as unary fuzzy connective

Note that the Fredkin gate is a linear operator on $\otimes^{n+m+1}\mathbb{C}^2$. It allows us to conceive the Fredkin gate even as a unary connective. However, by Equation (4), the propability value $p(F^{(n,m,1)}(\rho))$ intrinsecally encloses a fuzzy expression originated by the reduced states of ρ. In this section we deal with this argument.

In the rest of the section, by Equation (4) if ρ is a density operator on $\otimes^{n+m+1}\mathbb{C}^2$ we shall consider the decomposition

$$\rho = \rho_{n+m} \otimes \rho_1 + M(\rho),$$

where ρ_{n+m} and ρ_1 are the reduced states of ρ with respect to the subspaces $\otimes^{n+m}\mathbb{C}^2$ and $\mathbb{C}^2$, respectively.

THEOREM 8.1
Let ρ be a density operator on $\otimes^{n+m+1}\mathbb{C}^2$, where $diag(\rho) = (r_i)_{i=1}^{2^{n+m+1}}$. Then,

$$p(\mathbb{F}^{(n,m,1)}(\rho)) = (1-p(\rho_1))\alpha^{m,n}(\rho_{n+m}) + p(\rho_1)(1-\beta^{m,n}(\rho_{n+m})) + p(\mathbb{F}^{(n,m,1)}(M(\rho)),$$

and

$$p(\mathbb{F}^{(n,m,1)}(M(\rho)) = \gamma^{n,m+1}(\rho) + \sum_{i=0}^{2^m-1}\sum_{j=1}^{2^{n-1}} r_{(2j-1)2^{m+1}+3+4i} + r_{(2j-1)2^{m+1}+4+4i}$$
$$-(1-p(\rho_1))\alpha^{m,n}(\rho_{m+n}) + p(\rho_1)(1-\beta^{m,n}(\rho_{m+n})),$$

being

$$\alpha^{m,n}(\rho_{m+n}) = \sum_{i=1}^{2^{m-1}}\sum_{j=0}^{2^{n-1}-1}[r_{2i2^{n+1}+3+4j} + r_{2i2^{n+1}+4+4j}],$$

$$\beta^{m,n}(\rho_{m+n}) = \sum_{i=1}^{2^{m-1}}\sum_{j=0}^{2^{n-1}-1}[r_{2i2^{n+1}+1+4j} + r_{2i2^{n+1}+2+4j}].$$

PROOF. By Theorem 7.1 we have that

$$\begin{aligned} p(\mathbb{F}^{(n,m,1)}(\rho)) &= p(\mathbb{F}(\rho_{n+m} \otimes \rho_1)) + M \\ &= (1-p(\rho_1))\alpha^{m,n}(\rho_{n+m}) + p(\rho_1)(1-\beta^{m,n}(\rho_{n+m})) + M. \end{aligned}$$

Let us consider the Fredkin $F^{(n,m,1)}$ applied to ρ on $\otimes^{n+m+1}\mathbb{C}^2$, where

$$\begin{aligned} diag(\rho) = & [|(r_1,\cdots,r_{2^{m+1}}),(r_{2^{m+1}+1},\cdots,r_{2(2^{m+1})})|,\\ & \cdots |(r_{2(2^{m+1}+1)},\cdots,r_{3(2^{m+1})}),(r_{3(2^{m+1})+1},\cdots,r_{4(2^{m+1})})|,\\ & \vdots \\ & \cdots |(r_{(2^n-2)2^{m+1}+1},\cdots,r_{(2^n-1)2^{m+1}}),(|(r_{(2^n-1)2^{m+1}+1},r_{2^m(2^{n+1})})]. \end{aligned}$$

Then, by recalling the matrix representation of the Fredkin gate given in Equation (6) and through a straightforward calculation, we can see that

$$p(\mathbb{F}^{(n,m,1)}(\rho))=\gamma^{n,m+1}(\rho)+\sum_{i=0}^{2^m-1}\sum_{j=1}^{2^{n-1}}[r_{(2j-1)2^{m+1}+3+4i}+r_{(2j-1)2^{m+1}+4+4i}].$$

On the other hand, by Equation (4) and by Theorem 7.1, we have that

$$\begin{aligned} p(\mathbb{F}^{(n,m,1)}(\rho))=(\mathbb{F}^{(n,m,1)}(\rho_{m+n}\otimes\rho_1+M_1(\rho)))=\\ =p(\mathbb{F}^{(n,m,1)}(\rho_{m+n}\otimes\rho_1))+p(\mathbb{F}^{(n,m,1)}(M_1(\rho)))=\\ (1-p(\rho_1))\alpha^{m,n}(\rho_{m+n})+p(\rho_1)(1-\beta^{m,n}(\rho_{m+n}))+p(\mathbb{F}^{(n,m,1)}(M_1(\rho))). \end{aligned}$$

Hence,

$$\begin{aligned} p(\mathbb{F}^{(n,m,1)}(M_1(\rho))) = & \gamma^{n,m+1}(\rho)\\ & +\sum_{i=0}^{2^{m-1}-1}\sum_{k=0}^{2^{n-1}-1} r_{(2i+1)2^{m+1}+3+4k}+r_{(2i+1)2^{m+1}+4+4k}\\ & -(1-p(\rho_1))\alpha^{m,n}(\rho_{m+n})+p(\rho_1)(1-\beta^{m,n}(\rho_{m+n})), \end{aligned}$$

where

$$\alpha^{m,n}(\rho_{m+n})=\sum_{i=1}^{2^{m-1}}\sum_{j=0}^{2^{n-1}-1}[r_{2i2^{n+1}+3+4j}+r_{2i2^{n+1}+4+4j}]$$

and

$$\beta^{m,n}(\rho_{m+n})=\sum_{i=1}^{2^{m-1}}\sum_{j=0}^{2^{n-1}-1}[r_{2i2^{n+1}+1+4j}+r_{2i2^{n+1}+2+4j}].$$

■

To point out the fuzziness of $p(\mathbb{F}^{(m,n,1)}(\rho))$ by Theorem 7.2 we have that

$$\begin{aligned} p(\mathbb{F}^{(n,m,1)}(\rho)) &= p(\mathbb{F}(\rho_{n+m}\otimes\rho_1))+p(\mathbb{F}^{(n,m,1)}(M(\rho))=\\ &= \neg p(\rho_n)\cdot p(\rho_1)\oplus p(\rho_n)\cdot p(\rho_m)+p(\mathbb{F}^{(n,m,1)}(M(\rho)), \end{aligned}$$

where ρ_n, ρ_m and ρ_1 are the reduced states of ρ with respect to the subspaces $\otimes^n\mathbb{C}^2$, $\otimes^m\mathbb{C}^2$ and $\mathbb{C}^2$, respectively. We also note that as well as ρ_{n+m} is the reduced state of ρ with respect to $\otimes^{n+m}\mathbb{C}^2$, analogously ρ^n and ρ^m are the reduced states of ρ_{n+m} with respect to the subspaces $\otimes^n\mathbb{C}^2$ and $\otimes^m\mathbb{C}^2$, respectively. Thus the probability value $p(\mathbb{F}^{(n,m,1)}(\rho))$ can be expressed as a sum of a PMV-term: the fuzzy component depending on the reduces states of ρ and the quantity $p(\mathbb{F}^{(n,m,1)}(M(\rho))$ arising from the non-factorizability of the quantum state ρ. This representation allows to conceive the Frendkin gate as a kind of unary connective.

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