

# Pattern Recognition in Non-Kolmogorovian Structures

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**Abstract** We present a generalization of the problem of pattern recognition to arbitrary probabilistic models. This version deals with the problem of recognizing an individual pattern among a family of different species or classes of objects which obey probabilistic laws which do not comply with Kolmogorov's axioms. We show that such a scenario accommodates many important examples, and in particular, we provide a rigorous definition of the classical and the quantum pattern recognition problems, respectively. Our framework allows for the introduction of non-trivial correlations (as entanglement or discord) between the different species involved, opening the door to a new way of harnessing these physical resources for solving pattern recognition problems. Finally, we present some examples and discuss the computational complexity of the quantum pattern recognition problem, showing that the most important quantum computation algorithms can be described as non-Kolmogorovian pattern recognition problems.

**Keywords** Quantum pattern recognition · Quantum algorithms · Convex operational models

## 1 Introduction

Pattern recognition is an active field of research which has many applications in different disciplines, such as information science, economics, engineering, and machine learning (Bishop 2006; Chen and Chen 2016). Intuitively, pattern recognition could be defined as the problem of how a rational agent (which could be an automata), decides to which class

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of objects a given new object belongs. In its simpler version, given a family of classes of objects  $C_i$  (representing objects of different kinds), the rational agent must decide to which class a given object  $a$  belongs. The comparison is made with regards to a given set of properties  $\alpha_j$  of  $a$ . More sophisticated versions assume that the knowledge of the rational agent is given in terms of probability distributions. In its classical version, the properties involved are compatible (can have definite simultaneous values) and probabilities are Kolmogorovian (in the sense that they can be described by the well known Kolmogorov's 1933 axioms).

But it may happen that, for some particular models, the properties involved cannot be determined simultaneously. This could be the case if, for example, there is a limitation in the capability of acquisition of knowledge by the agent. This could be originated in epistemic constraints (for example, in game theory), or in ontological limitations (as in quantum mechanics). Non-classical effects have also been observed in cognitive phenomena (see for example Aerts et al. 2015; Aerts and de Bianchi 2015). More concretely, suppose that our rational agent deals with probabilistic models which do not obey the laws of classical physics, in the sense that it is not possible to attribute simultaneous values to the properties of the objects involved (Döring 2005; Kochen and Specker 1967). How can we formulate the pattern recognition problem in this non-Boolean (Al-Adilee and Nánásiová 2009; Svozil 2009) framework? The theory of pattern recognition for this case cannot be the same as before, essentially, due to the non-commutative character of the properties and the probabilities involved. In other words, complementarity poses a problem for the treatment of objects as possessing simultaneous collections of well defined properties. For example, in the quantum case, we must acknowledge the fact that the best way of describing a class of objects is by attributing probabilities governed by the laws of quantum mechanics (Gudder 1979; Rédei 1998). Then, observable quantities will be represented mathematically by (possibly) non-commutative operators acting on a Hilbert space (von Neumann 1996), and this gives a different formulation of the discrimination problem (for concrete examples of this, see Sentís et al. 2015; Guță and Kotlowski 2010; see also Monràs et al. 2016 for more discussion).

In order to describe how things work when probabilities depart from the Kolmogorovian case, we present a formal quantum patterns recognition's framework for generalized probabilistic models (Barnum et al. 2007, 2013; Barnum and Wilce 2011). In this way we considerably expand the domain of applicability of this field of research. Our main aim is to focus attention on the fact that there are several versions of the problem, depending on the structural aspects of the probabilities involved. Our theoretical framework allows for introducing rigorous definitions of the classical and quantum pattern recognition problems. At the same time, it allows one to envisage the existence of other versions of the problem, as it would appear, for example, in the relativistic or thermodynamic limits.

It is important to remark that there are other approaches that use non-classical techniques or quantum systems (like quantum computers) to solve pattern recognition problems (see for example Horn and Assaf 2002; Schuld et al. 2014; Schützhold 2003; Trugenberger 2002; Sergioli et al.). But they differ from our approach, mainly because the entities to be discerned are classical (i.e., they do not exhibit quantum phenomena such as superposition or entanglement). There also exist previous formulations of the problem which are similar to ours for the particular case of non-relativistic quantum mechanics and quantum optics (see for example Aïmeur et al. 2006; Guță and Kotlowski 2010; Monràs et al. 2016; Sasaki and Carlini 2002; Sasaki et al. 2001; Sentís et al. 2015), that can be naturally accommodated into our more general framework. Our generalization could be useful for a better

understanding of these models, and, at the same time, it could serve as a suitable tool for describing more general physical situations.

This perspective opens a field of research which is richer (from a physical point of view) than its previous classical versions, mainly because we allow for the possibility that the different classes of objects involved display non-classical features, such as complementarity, or non-classical correlations (such as entanglement or discord). Things may change in a subtle manner when these non-classical features cannot be neglected. This behavior would be expected in any situation in which the elements involved in the analysis are “small” enough and reach the molecular or atomic level. Of course, our approach can also be useful for classical systems which are structured in such a way that, for one reason or another, exhibit features that imitate quantum phenomena (this is the case in some examples of game theory). Remarkably enough, a connection between the study of relational databases and the violation of Bell’s inequalities is presented in Abramsky (2013). This study suggests that some mathematical structures underlying quantum contextuality can be found in fields of research in which the data we might be dealing with is not necessarily about physical objects. In this way, future developments of our mathematical framework could be of use for the study of problems outside of physics, such as relational databases and Big Data.

The paper is organized as follows. In Sect. 2 we review the standard approach to the pattern recognition problem. Next, in Sect. 3, we review the different mathematical frameworks which allow us to represent probabilities and properties which depart from the Boolean–Commutative case. In Sect. 4 we present our version of the pattern recognition problem for generalized probabilistic models, and show how non-classical correlations can appear in the non-Kolmogorovian probabilistic setting. In Sect. 5 we describe the particular cases of the standard, relativistic and statistical quantum mechanical settings as concrete examples. We study possible connections between quantum pattern recognition theory and some relevant quantum algorithms in Sect. 6. Finally, in Sect. 7 we draw some conclusions.

## 2 Classical Pattern Recognition Problem

Suppose that, given a family of different classes  $C_i$ , a rational agent (it could be a person or an automata) must decide to which class a given individual  $X$  pertains. It is important to remark that the classes could be disjoint or not. For example, if  $C_1$  is a collection of dogs and  $C_2$  is a collection of cats, the aim of the agent is to decide, given an unknown individual  $X$ , if it is a cat or a dog (from now on, following the jargon commonly used in pattern recognition, we use the terms “object” and “individual” interchangeably). It is usually assumed that knowledge about the different classes and the given individual is given in terms of a particular collection of properties (also called *features*) of all possible individuals in question. The collection of properties of a given individual  $X_j^i \in C_i$  (individual  $j$  belonging to class  $C_i$ ) is represented by an  $n$ -vector  $\alpha_j^i = (\alpha_1^{ij}, \alpha_2^{ij}, \dots, \alpha_n^{ij})$  (where the  $\alpha_k^{ij}$  take real values) and probability distributions representing degrees of belief of the agent regarding each individual having a particular collection of properties. This is the most elementary form of the problem of *pattern recognition* (Bishop 2006). We will refer to it as the *classical pattern recognition problem*.

To be more specific, suppose that, given an individual  $X$  to be recognized, knowledge about it is represented by a probability distribution  $p(\alpha)$ . Suppose also that knowledge

about each class  $C_i$  is represented by a probability distribution  $p_i(\alpha_j^i)$  assigning a weight to each property vector  $\alpha_j^i$  in  $C_i$ . Thus, the *classification problem*, is the problem of determining to which class the individual is assigned to by contrasting knowledge about the individual and the different classes. This can be done, for example, by comparing the mean values of the considered properties using a suitable measure (or by directly comparing the different probabilities involved). The output will be a probabilistic assertion of the form “ $x$  is assigned to the class  $C_i$  with probability  $p(C_i)$ ” (in other words, the output will be a vector  $(p(C_1), \dots, p(C_m))$ , with complete certainty when a particular  $p(C_i)$  is one and the rest is zero). When probabilities are involved, we will refer to it as the *probabilistic classical pattern recognition problem*. If there are no probabilities involved, we will say that the problem is *deterministic*.

### 3 Non-Kolmogorovian Measures and Convex Operational Models

In the above formulation of the pattern recognition problem, it is assumed that the properties involved are classical. In other words, the objects are assumed to possess a collection of properties which can assume definite values at the same time. But if the objects involved obey the laws of quantum mechanics, then, incompatible properties may come into play. As it is well known, the Kochen–Specker theorem precludes the possibility of assigning states defining simultaneous definite properties to quantum systems (see for example Kochen and Specker 1967 for standard quantum mechanics and Döring 2005 for general von Neumann algebras). Consequently, the probabilistic measures involved will no longer be Kolmogorovian (Rédei and Summers 2007): probabilities are now assigned to elements in the orthomodular lattice of projection operators in a Hilbert space (von Neumann 1996). In the following we review the formal structure of these non-classical features.

A suitable framework to begin the study of non-Kolmogorovian probabilities is that of measures in *orthomodular lattices* (Al-Adilee and Nánásiová 2009; Beltrametti and Cassinelli 1981; Hamhalter 2003; Kalmbach 1983; Ledda and Sergioli 2010) (see also Aerts 2002, for a formulation of non-Kolmogorovian probabilities as functions valued in subsets instead of numbers). Suppose that an algebra of events can be represented as an orthomodular lattice  $\mathcal{L}$ .<sup>1</sup> Then, a generalized probabilistic measure can be represented as a function (Dalla Chiara et al. 2004):

$$v : \mathcal{L} \rightarrow [0, 1],$$

such that

$$v(\mathbf{1}) = 1,$$

and, for a denumerable and pairwise orthogonal family of events

$$v\left(\sum_{i \in I} E_i\right) = \sum_{i \in I} v(E_i). \tag{1}$$

When  $\mathcal{L}$  is a Boolean algebra, the above axioms reduce to the well known Kolmogorov’s

<sup>1</sup> An *orthomodular lattice*  $\mathcal{L}$ , is an orthocomplemented lattice satisfying that for any  $a, b$  and  $c$ , if  $a \leq c$ , then  $a \vee (a^\perp \wedge c) = c$ . We refer the reader to Kalmbach (1983) for a detailed exposition.

axioms for classical probability calculus. In this framework, the elements of a Boolean algebra are intended to represent properties of a classical system. On the other hand, if  $\mathcal{L}$  represents the orthomodular lattice of projection operators acting on the Hilbert space of a quantum system,<sup>2</sup> we recover—via the celebrated Gleason’s theorem (Buhagiar et al. 2009; Gleason 1957)—the probability assignment given by Born’s rule: if a quantum system is prepared in state  $\rho$ , the probability of observing the property represented by projection  $P$  is given by

$$p_\rho(P) = \text{tr}(\rho P). \tag{2}$$

Typically, more general theories of interest can be considered, as for example, the projection lattices of algebraic relativistic quantum field theory, which involves projection lattices of Type III factors (Rédei and Summers 2007), or algebraic quantum statistical mechanics (Rédei and Summers 2007). We want to stress that the underlying algebraic structure determines the structural features of the probabilities and correlations involved, and this defines essentially different pattern recognition models.

We will discuss some examples of this in the following sections. As is well known, observables in quantum mechanics can be represented as operators in a Hilbert space, being a non-commutative algebra their most distinctive feature (von Neumann 1996). This is strongly related with the non-Booleanity of the lattice of projection operators (Beltrametti and Cassinelli 1981; von Neumann 1996). In the next section, we present a version of the pattern recognition problem for non-Kolmogorovian probabilistic measures which includes the quantum and classical cases as particular instances.

It is easy to show that the set of states defined by Eq. (1) is convex. This feature can be taken as the starting point for a more general approach to the study statistical theories, based on the geometrical properties of convex sets (Barnum et al. 2007, 2013; Barnum and Wilce 2011). Assume that the set of states of a given model is represented by a convex set  $\mathcal{S}$ . Then, for each observable with outcome set  $X$ , a given state  $s \in \mathcal{S}$  should define a probability  $p(s, x)$  for each possible outcome  $x \in X$ . Given a state  $s \in \mathcal{S}$  and any outcome  $x \in X$ , it is natural to define an affine evaluation-functional  $f_x : \mathcal{S} \rightarrow [0, 1]$  by  $f_x(s) := s(x)$  (where  $s(x)$  is a real number in the interval  $[0, 1]$  that represents the evaluation). Then, it is reasonable to consider each functional  $f : \mathcal{S} \rightarrow [0, 1]$  as representing a measurement outcome, and thus represent that outcome by  $f(s)$  (if the state of the system is  $s$ ). In this way, states are interpreted as points of a convex set, embedded in a vector space  $V(\mathcal{S})$ , and observables (called *effects* in the generalized setting) as continuous linear functionals in the dual space  $V^*(\mathcal{S})$  acting on this set. It turns out that the shape of the convex set has information about the model involved. For example, the faces of the convex set of a quantum system define an orthomodular lattice which is isomorphic to the lattice of projection operators, while for classical systems, the set of faces forms a Boolean lattice (Bengtsson and Życzkowski 2006). In this way, for some important models, it is possible to relate the approach based on measures over lattices with the approach based on convex sets. Notions like those of pure and mixed states, entanglement and information, are defined in a natural way, which generalizes the quantum scenario. We refer the reader to Barnum et al. (2007), Barnum et al. (2013) and Barnum and Wilce (2011) for more details.

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<sup>2</sup> In the Hilbert space case, projection operators are in one to one correspondence to closed subspaces (thus, these notions are interchangeable). Representing “ $\vee$ ” by the closure of the sum of two subspaces, “ $\wedge$ ” by its intersection, “ $(\dots)^\perp$ ” by taking the orthogonal complement of a given subspace and “ $\leq$ ” by subspace inclusion, it is possible to show that subspaces (and thus, projections) possess an orthomodular lattice structure.

## 4 Pattern Recognition in the Generalized Setting

Let us now introduce our general framework for dealing with the quantum pattern recognition problem in generalized probabilistic models. It will contain the quantum and classical versions of the problem as particular cases.

Given a collection of classes of objects  $O_i$ , let us assume that the state of each object  $o_j^i$  (i.e., object  $j$  of class  $O_i$ ) is represented by a state  $v_j^i \in \mathcal{C}_i$ , where  $\mathcal{C}_i$  is the convex operational model representing object  $o_j^i$ . We will assume that all objects in the class  $O_i$  are represented by the same convex operational model  $\mathcal{C}_i$  (i.e., they are all elements of the same type). Then, suppose that weights  $p_j^i$  are assigned to the objects  $o_j^i$ , representing the rational agent's knowledge about the importance of object  $o_j^i$  as a representative of class  $\mathcal{C}_i$  with respect to other objects in the class (if all objects are equally important, the weights are chosen as  $p_j^i = \frac{1}{N_i}$ , where  $N_i$  is the total number of objects—i.e., the cardinality—of the class  $\mathcal{C}_i$ ). This means that the probabilistic state of the whole class  $O_i$  can be represented by a mixture  $v_i = \sum_j p_j^i v_j^i \in \mathcal{C}_i$ . As we discuss below, it is also possible to assume that non-local correlations are given between the different classes, and the states  $v_i$  are reduced states of a global—possibly entangled—state  $\tilde{v}$ . But we notice that under these conditions, the states  $v_i$  will be *improper mixtures*, and then, no consistent ignorance interpretation can be given for them (D'Espagnat 1976).

The generalized pattern recognition problem is then posed as follows. A particular object  $o$  must be identified and compared with the information given by the generalized states of the classes represented by  $v_i$  (or more generally, by  $\tilde{v}$ ), obtained in the learning process. The comparison could be also restricted to a collection of properties  $\mathbf{a} = (\alpha_1, \dots, \alpha_m)$ , represented now by generalized effects  $\alpha_i$ . We will assume, as usual, that knowledge about  $o$  is represented by a generalized state  $v$ . Notice that, in order to obtain  $v$ , several copies of the unknown object  $o$  may be needed, whenever the probabilistic character of the model is irreducible. This is the case in quantum mechanics: if more copies are available, the reconstruction of the state of the unknown object will be more accurate, and this can be used to improve the classification process.

Different techniques for discriminating the given state with regard to the states of the classes were studied for some particular models (see for example Guță and Kotłowski 2010; Sentís et al. 2015). Notice that optimal classification strategies may depend strongly on the structural properties of the probabilities associated to the model involved. Notice also that non-classical correlations between the classes represented by states  $v_i$  may come into play (the  $v_i$  may be reduced states of a global state  $\mu$ ). Again, the particularities of the correlations originated in each model may be critical here (see for example Clifton and Halvorson 2001, for a discussion of the differences between relativistic and non-relativistic quantum entanglement).

It is easy to see that, if the  $\mathcal{C}_i$ 's are simplices (hyper-tetrahedrons), then, we will recover the classical problem of patterns' recognition, discussed in the introduction. Indeed, probabilities on simplices are isomorphic to measures over Boolean algebras, and thus, we have Kolmogorovian probabilities. And this is nothing but our definition of classical pattern recognition problem. As it is well known, simplices admit dispersion-free states: this means that using this description, we can also recover the deterministic version of the problem described in Sect. 2. On the other hand, as we remarked in Sect. 3, the sets of states of quantum systems are naturally convex sets, but are not simplices (Bengtsson and Życzkowski 2006).

## 5 Examples

As examples of the general framework introduced above, in this section we briefly describe non-classical examples of the pattern recognition problem which originate in non-equivalent physical theories of interest.

### 5.1 Quantum Pattern Recognition

Suppose that we are given a collection of quantum objects each belonging to a particular class  $Q_i$ , and given a particular object  $q$ , the rational agent aims to determine to which class it is assigned. We look now for a quantal version of the problem posed in Sect. 2. First, we must assume, as the most general possibility, that the collection of chosen properties can be non-commutative. Thus, the properties of object  $q_j^i$  (object  $j$  of class  $C_i$ ) will be represented by operators<sup>3</sup> acting on a Hilbert space  $\mathcal{H}_i$  (representing the class  $Q_i$ ). It is now impossible (in the general case) to assign a vector of definite properties to each object, due to possibly non-commutativity of the operators involved. The only thing that we can do, is to assign probabilities for each property coordinate using the quantum state  $\rho_j^i$  of each object  $q_j^i$ . Thus, if—as in the classical case—we assign weights  $p_j^i$  to each object  $q_j^i$ , knowledge about the class  $Q_i$  can now be represented by a mixture  $\rho_i = \sum_j p_j^i \rho_j^i$ . Given the fact that in general, interaction between physical systems represented by classes  $Q_i$  can be non-negligible, and thus, non-trivial correlations may be involved, we will assume that the states  $\rho_i$  are arbitrary states of the Hilbert space  $\mathcal{H}_i$  (i.e., the  $\rho_i$  are not necessarily proper mixtures). We call  $\tilde{\rho}$  the global state of the whole set of classes.

Given an arbitrary individual  $q$ , we are thus faced with the problem of determining to which class  $Q_i$  it should be assigned. In the general case, the state of  $q$  will be represented by a density operator  $\rho$  (acting on one of the unknown Hilbert spaces  $\mathcal{H}_i$ , but certainly embedded in the Hilbert space  $\mathcal{H}_1 \otimes \mathcal{H}_2 \cdots \otimes \mathcal{H}_n$ ). Notice however, that the state  $\rho$  could, in the general case, be unknown to the agent, and he may have only access to a sample of values  $\{a_j\}$  of the operators  $\sigma_j$ . Thus, for the classification problem, he should be able to, either reconstruct the unknown state  $\rho$  using quantum statistical inference methods, or just directly compare the sampled values with the information provided by the global state  $\tilde{\rho}$ .

Our version of the quantum pattern recognition problem can be interpreted in a very direct physical way. What we have shown in this section is that if the objects involved exhibit non-classical features (and this could be the case each time that the systems involved are small enough to exhibit quantum behavior), then, the rational agent will be confronted with complementarity phenomena, non-Kolmogorovian probability measures, and non-classical correlations. In this way, the information about the parties involved must be necessarily represented by density operators. Furthermore, as we have seen, the classification problem must be adapted to this situation in such a way that quantum statistical inference techniques must be used in order to decide which will be the class to which the object will be most likely assigned.

From this physical perspective, how is it possible to represent information updating and learning? In other words, which is the quantum analogous of a semi-supervised system? This can be suitably described using quantum operations as follows. Suppose now that at an initial state, the agent has an information  $\rho_i(0)$  for each class  $Q_i$ , and he is confronted with an individual of which it has information  $\rho(0)$ , and a global state  $\tilde{\rho}(0)$ . Then, after the classification process at time  $t$ , it is necessary to update knowledge about the classes and the global

<sup>3</sup> Notice that these operators could be quantum effects without loss of generality.

state to new states  $\rho_i(t)$  and  $\tilde{\rho}(t)$ , respectively. This can be suitably modeled by a quantum operation  $A(t)$  acting on the convex quantum set of states of  $\mathcal{C}(\mathcal{H}_1 \otimes \mathcal{H}_2 \cdots \otimes \mathcal{H}_n)$ , such that  $A(t)\tilde{\rho}(0) = \tilde{\rho}(t)$ . A *quantum learning operator* will be thus a family of quantum operations  $\{A(t_1), \dots, A(t_n)\}$ . Hence, a *quantum learning process* will be a succession of global states  $\{\tilde{\rho}(0), A(t_1)\tilde{\rho}(0), A(t_2)\tilde{\rho}(t_1), \dots, A(t_n)\tilde{\rho}(t_{n-1})\}$ . The goal of the learning process will be achieved if the uncertainty of the final state is reduced. The dispersion could be measured using the von Neumann entropy (or other quantum entropic measures) (Bosyk et al. 2016; Holik et al. 2016, 2015).

This dynamical view of the quantum learning process can be easily generalized to arbitrary statistical models by appealing to affine maps. The entropies used to measure the success of the learning process could be the Measurement Entropy (or any other entropic measure of interest which can be suitably generalized Holik et al. 2016, 2015).

Let us also notice that in the case of unsupervised learning—where the training data are not labeled—the classes of the classification process are, in principle, unknown and the learning process is fundamental in order to individuate the distribution of these classes. If the assignment of each object to its respective class is given by a Kolmogorovian probability function, then the distribution of the classes that arises from the learning process is a kind of partition such that each member of the dataset is classified within one and only one class; the Voronoi diagram is an instance of this kind of partition. On the other hand, if the probabilities involved to describe the objects in question are non-Kolmogorovian, the distribution of the classes that arises from the learning process has to take into account this change in the mathematical description. For the quantum case, the mathematical description of the states of the objects in the dataset is given in terms of vectors in a Hilbert space (or more generally, density operators), while their properties are represented by self-adjoint operators (see Aïmeur et al. 2006).

## 5.2 Pattern Recognition in ARQFT

In algebraic relativistic quantum field theory (ARQFT), a  $C^*$ -algebra is assigned to any open set  $O$  of a differential manifold  $M$  (Halvorson and Müger 2006; Haag 1996). Open sets are intended to represent local regions, and  $M$  models space-time with its symmetries. Local algebras are intended to represent local observables (such as particle detectors). For example, in ARQFT,  $M$  is Minkowski's four dimensional space-time, endowed with the Poincaré group of transformations.

It turns out, that (global) states of the field define measures over the local algebras. But in general, the local algebras of ARQFT will not be Type I factors as in standard quantum mechanics. For example, it can be shown that for a diamond region, a Type III factor must be assigned (Yngvason 2005). This means that the orthomodular lattice involved in axioms 1, will not be the lattice of projection operators of a Hilbert space, but a one with different properties. For a discussion on the properties of lattices associated to von Neumann algebras see Sect. 6.2 (Rédei 1998). Consequently, operator algebras in ARQFT are quite different to those of standard quantum mechanics. This is expressed, for example, in the properties of correlations (see for example Clifton and Halvorson 2001 and Chapter 3 in Halvorson and Müger 2006).

This means that the discrimination problem must be posed between classes  $F_i$  represented by states of the field  $\varphi_i$  and a given individual state  $\varphi$ .<sup>4</sup> As far as we know, this

<sup>4</sup> In practical implementations, these states and the discrimination problem, could be restricted to a concrete space-time region.

problem was not addressed in the literature from the point of view of pattern recognition. But it is an important one, because in the general case, it could be useful for information protocols based on quantum optics (where the effects of the field character of the theory cannot be neglected). In particular, a simpler but analogous version of the problem could be conceived by appealing to the Fock-space formalism, in order to describe the fields and the states involved (see for example Sentís et al. 2015, for a version of the problem posed in terms of coherent states of light using the Fock-space formalism).

### 5.3 Pattern Recognition in AQSM

As in the quantum field theoretic example, a similar problem can be posed in the algebraic approach to quantum statistics (AQSM). Here, a typical problem could be to discern a kind of atoms from a set of classes of gasses; now, the comparison will be between the state of the item and the classes involved. But it can be shown that the global states of a gas, as described by AQSM, will be in general, a measure over a factor different from the Type I case (see for example Bratteli and Robinson 1997, Chapter 5 and Rédei and Summers 2007). This means that, again, the pattern recognition problem will depart from that of the classical one, but also from that of standard quantum mechanics (where we have Type I projection lattices).

An example of interest could appear in problems related to image recognition. To clarify ideas, let discuss first a classical version of the problem. Suppose that a machine has to solve a problem of recognizing handwritten digits. These drawings are first transformed into digitalized images of  $n \times n$  pixels. This means that the information of each image is stored in a vector  $\mathbf{x}$  of length  $n \times n$ . The goal is to build our automata in such a way that it takes a vector  $\mathbf{x}$  as an input, and gives us as output the identity of the digit in question (Bishop 2006). Now we pose the question: in a real hardware, this vector should be stored using bits of a given length. But if the components of the hardware are so small that they become quantal (as in Yang et al. 2016), then, we may have a large chain of qubits used to store this information. This means that the information will be stored now in a potentially large chain of qubits  $\bigotimes_{i=1}^n \mathbb{C}_i^2$ . When  $n$  is big enough, the number of particles involved to store the information may become statistical. Thus, the approximation of  $n \rightarrow \infty$  becomes more and more realistic in order to represent global properties of the information stored. But in this limit, we need the algebras corresponding to the algebraic formulation of quantum statistical mechanics.

Let us illustrate this with a concrete model. Suppose that we have a spacial arrangement  $L$  of  $N$ -dimensional quantum systems. For each point  $x \in L$  we have a Hilbert space  $\mathcal{H}_x$ , and for each subset of points  $\Gamma \in L$ , the associated Hilbert space is given by the tensor product  $\mathcal{H}_\Gamma = \bigotimes_{x \in \Gamma} \mathcal{H}_x$ . Thus, every subset  $\Gamma \in L$  has associated an algebra  $\mathcal{A}(\mathcal{H}_\Gamma)$ . The norm completion of the collection  $\mathcal{A} = \{\mathcal{A}_\Gamma\}_{\Gamma \in L}$  is a quasi-local  $C^*$ -algebra when equipped with the net of  $C^*$ -subalgebras  $\mathcal{A}_\Gamma$ . Thus, the classification problem must be done with respect to states defined in this algebra (such as KMS-states Bratteli and Robinson 1997), whose properties are different to that of a Type I factor.

## 6 Quantum Algorithms as Quantum Pattern Recognition Problems

Recent developments suggest that quantum speedups appear in structured problems (Aaronson and Ambainis 2014): the problem must exhibit some structure or pattern in order that the quantum computer display an overhead with respect to a classical one.

Indeed, in Sasaki et al. (2001) the authors suggest that in a certain sense, the most important quantum computation algorithms can be viewed as pattern recognition problems. Let us now outline how our generalized formalism could be useful to formulate these deep intuitions on a more solid ground, by looking at some examples of quantum algorithms.

### 6.1 Deutsch–Jozsa algorithm

Let us examine first the Deutsch–Jozsa algorithm (Nielsen and Chuang 2010). In this case, the task is to determine if a function  $f$  is constant or balanced. There are four functions from  $\{0, 1\} \rightarrow \{0, 1\}$ , namely:

$$\begin{aligned} f_1(0) = 0 \quad f_1(1) = 1 \\ f_2(0) = 1 \quad f_2(1) = 0 \\ f_3(0) = 0 \quad f_3(1) = 0 \\ f_4(0) = 1 \quad f_4(1) = 1 \end{aligned} \tag{3}$$

Thus, we have two classes:  $C = \{f_1, f_2\}$  and  $B = \{f_3, f_4\}$ , and we must determine if the function  $f$  belongs to  $B$  or to  $C$ . Up to now, this is just a classical pattern recognition problem.

Let us see now how the quantum computer transforms this problem into a quantum pattern recognition one. The computer is prepared first in the quantum state  $|0\rangle|1\rangle$ . Next, the Hadamard operator is applied to both qubits yielding the state:

$$\frac{1}{2}(|0\rangle + |1\rangle)(|0\rangle - |1\rangle).$$

Next, the quantum implementation of the function  $f$  (which brings the connection between the classical problem and the quantum computation) will be given by a quantum operator such that it maps  $|x\rangle|y\rangle$  to  $|x\rangle|f(x) \oplus y\rangle$ . Applying this function to the state gives

$$(-1)^{f(0)} \frac{1}{2} \left( |0\rangle + (-1)^{f(0) \oplus f(1)} |1\rangle \right) (|0\rangle - |1\rangle).$$

Now, applying the Hadamard transformation again to the first qubit we get:

$$|\psi\rangle = (-1)^{f(0)} \frac{1}{2} \left( \left( 1 + (-1)^{f(0) \oplus f(1)} \right) |0\rangle + \left( 1 - (-1)^{f(0) \oplus f(1)} \right) |1\rangle \right) (|0\rangle - |1\rangle).$$

The next step consists in determining the projection of the above state to the subspaces represented by projection operators  $|0\rangle\langle 0| \otimes \mathbf{1}$  and  $|1\rangle\langle 1| \otimes \mathbf{1}$ . This is nothing but determining if the system represented by state  $|\psi\rangle\langle\psi|$  belongs to the classes represented by projections  $|0\rangle\langle 0| \otimes \mathbf{1}$  and  $|1\rangle\langle 1| \otimes \mathbf{1}$ : the computation of the projections is nothing but the Hilbert Schmidt distance between these operators. Thus, this simple problem shows that this is a pattern recognition problem in which the rational agent has to decide if an individual (the output state of the computer previous to measurement) represented by state  $\rho = |\psi\rangle\langle\psi|$  belongs to the class represented by state  $\rho_1 = |0\rangle\langle 0| \otimes \mathbf{1}$  and the class represented by state  $|1\rangle\langle 1| \otimes \mathbf{1}$ .

### 6.2 Period of a Function Determination

The determination of the period of a periodic function  $f$  lies at the heart of the Shor and Simon quantum computation algorithms (Nielsen and Chuang 2010). Here we show that this problem can be reduced to a quantum pattern recognition one.

The objective now is to determine the period of a function  $f : \mathcal{Z}_N \rightarrow \mathcal{Z}$ , such that  $f(x + r) = f(x)$  for all  $x$ . It is assumed that the function does not take the same value twice in the same period. Start the computer as usual by generating the state:

$$|f\rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |x\rangle |f(x)\rangle. \tag{4}$$

It is not possible to extract the period yet. Even if we measure the value of the second register and obtain the value  $y_0$ , we will end up with the following state in the first register (with  $x_0$  the smallest  $x$  such that  $f(x) = y_0$  and  $N = Kr$ ):

$$|\psi\rangle = \frac{1}{\sqrt{K}} \sum_{k=0}^{K-1} |x_0 + kr\rangle. \tag{5}$$

But  $|\psi\rangle$  does not give us information about  $r$  yet. To do that, it is necessary to apply the quantum Fourier transform (QFT), which is a unitary matrix with entries

$$\mathcal{F}_{ab} = \frac{1}{\sqrt{N}} \exp^{2\pi i \frac{ab}{N}}. \tag{6}$$

By applying the QFT to  $|\psi\rangle$  we obtain

$$\mathcal{F}|\psi\rangle = \frac{1}{\sqrt{r}} \sum_{j=0}^{r-1} \exp^{2\pi i \frac{x_0 j}{r}} \left| j \frac{N}{r} \right\rangle. \tag{7}$$

Now, a measurement is performed in the basis  $\{|j \frac{N}{r}\rangle\}$ , and using the result it is possible to determine the period of the function as follows. The obtained value  $c$  will be such that  $c = j \frac{N}{r}$ , for some  $0 \leq j \leq r - 1$ . Then,  $\frac{c}{N} = \frac{j}{r}$ , and if  $j$  is coprime with  $r$ , it will be possible to determine  $r$ . The success of the algorithm depends on the fact that  $j$  and  $r$  will be coprimes with a high enough probability.

The key observation here, is that this can be cast as a pattern recognition problem as stated below. The objective is to decide to which class pertains an individual (again, the output of the computer after the second register measurement and application of the quantum Fourier transform) represented by state  $\mathcal{F}|\psi\rangle\langle\psi|\mathcal{F}$  (with  $|\psi\rangle = \frac{1}{\sqrt{K}} \sum_{k=0}^{K-1} |x_0 + kr\rangle$ ). The classes are represented by states  $\{\rho_i\}$ , with  $\rho_i = |j \frac{N}{r}\rangle\langle j \frac{N}{r}|$ . From the expression  $\text{tr}(\mathcal{F}|\psi\rangle\langle\psi|\mathcal{F}|j_0 \frac{N}{r}\rangle\langle j_0 \frac{N}{r}|)$ , which is the same as  $\text{tr}(|\psi\rangle\langle\psi|\mathcal{F}|j_0 \frac{N}{r}\rangle\langle j_0 \frac{N}{r}|\mathcal{F})$ , we obtain an equivalent problem by comparing state  $|\psi\rangle\langle\psi|$  with states  $\mathcal{F}|j_0 \frac{N}{r}\rangle\langle j_0 \frac{N}{r}|\mathcal{F}$ . We are of course interested in identifying those measurements for which  $\frac{c_0}{N}$  is an irreducible fraction.

## 7 Conclusions

In this paper we have presented a generalization of the pattern recognition problem to the non-commutative (or equivalently, non-Kolmogorovian) setting involving incompatible (non-simultaneously determinable) properties. In other words, we have cast the problem of pattern recognition for the case in which the state spaces involved are not simplexes. In this way, we have shown that it is possible to find some important (and non-equivalent) examples of interest: standard quantum mechanics, algebraic relativistic quantum field theory, and algebraic quantum statistics. The examples do not restrict only to these ones, but can include more general models, and particular, hybrid systems (classical and quantum). In particular, studies such as Abramsky (2013), suggest that the study of the pattern recognition problem in non-Kolmogorovian probabilistic models could, in principle, turn out to be particularly beneficial for the treatment of relational databases.

Next, we have shown that our perspective could be useful to characterize some of the most important quantum computation algorithms (Shor, Simon and Deutsch–Jozsa) as quantum pattern recognition problems. This may also suggest that it is to be expected that, translating classical pattern recognition problems into quantum ones, could lead to an improvement in the efficiency of the concomitant computation.

Our framework allows for clear definitions of the classical and quantum pattern recognition problems, respectively, and for an extension of the applicability of the problem to a wider domain.

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