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Entanglement and Quantum Logical Gates. Part II.

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Abstract We introduce the notion of *proper unitary connective-gate* and we prove that entanglement cannot be characterized by such gates. We consider then a larger class of gates (called *pseudo-unitary gates*), which contains both the unitary and the anti-unitary quantum operations. By using a *mixed* language (a proper extension of the standard quantum computational language), we show how a *logical characterization* of entanglement is possible in the framework of a *mixed* semantics, which generalizes both the unitary and the pseudo-unitary quantum computational semantics.

Keywords Entanglement · Quantum gates · Quantum logics

This article is dedicated to the memory of Peter Mittelstaedt. The authors will always remember the great scientist who has represented a noble example for all scholars of the IQSA-community.
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1 Proper Unitary Connective-gates

In the first part of this article we have given some negative and some partial positive answers to the general question "to what extent is a *logical characterization* of entanglement and of entanglement-measures possible?" We will now further investigate this question by considering other examples of gates that have a special logical interest.

It is expedient to recall the definition of some gates that will be used in this article. Let $\mathcal{H}^{(n)}$ be the Hilbert space $\underbrace{\mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2}_{n-times}$.

Definition 1 *(The negation)* For any $n \geq 1$, the *negation* (defined on $\mathcal{H}^{(n)}$) is the linear operator $\mathtt{NOT}^{(n)}$ such that, for every element $|x_1, \ldots, x_n\rangle$ of the canonical basis,

$$\mathtt{NOT}^{(n)}|x_1, \ldots, x_n\rangle = |x_1, \ldots, x_{n-1}\rangle \otimes |1 - x_n\rangle.$$

In particular, we obtain:

$$\mathtt{NOT}^{(1)}|0\rangle = |1\rangle; \quad \mathtt{NOT}^{(1)}|1\rangle = |0\rangle,$$

according to the classical truth-table of negation.

Definition 2 *(The Toffoli-gate)* For any $m, n, p \geq 1$, the *Toffoli-gate* (defined on $\mathcal{H}^{(m+n+p)}$) is the linear operator $\mathtt{T}^{(m,n,p)}$ such that, for every element $|x_1, \ldots, x_m\rangle \otimes |y_1, \ldots, y_n\rangle \otimes |z_1, \ldots, z_p\rangle$ of the canonical basis,

$$\begin{aligned}\mathtt{T}^{(m,n,p)}\ & |x_1, \ldots, x_m, y_1, \ldots, y_n, z_1, \ldots, z_p\rangle \\ & = |x_1, \ldots, x_m, y_1, \ldots, y_n, z_1, \ldots, z_{p-1}\rangle \otimes |x_m y_n \widehat{+} z_p\rangle,\end{aligned}$$

where $\widehat{+}$ represents the addition modulo 2.

Definition 3 *(The* XOR*-gate)* For any $m, n \geq 1$, the XOR-*gate* (defined on $\mathcal{H}^{(m+n)}$) is the linear operator $\mathtt{XOR}^{(m,n)}$ such that, for every element $|x_1, \ldots, x_m\rangle \otimes |y_1, \ldots, y_n\rangle$ of the canonical basis,

$$\mathtt{XOR}^{(m,n)}|x_1, \ldots, x_m, y_1, \ldots, y_n\rangle = |x_1, \ldots, x_m, y_1, \ldots, y_{n-1}\rangle \otimes |x_m \widehat{+} y_n\rangle.$$

Definition 4 *(The Hadamard-gate)* For any $n \geq 1$, the *Hadamard-gate* (defined on $\mathcal{H}^{(n)}$) is the linear operator $\sqrt{\mathtt{I}}^{(n)}$ such that for every element $|x_1, \ldots, x_n\rangle$ of the canonical basis:

$$\sqrt{\mathtt{I}}^{(n)}|x_1, \ldots, x_n\rangle = |x_1, \ldots, x_{n-1}\rangle \otimes \frac{1}{\sqrt{2}}\left((-1)^{x_n}|x_n\rangle + |1 - x_n\rangle\right).$$

In particular we obtain:

$$\sqrt{\mathtt{I}}^{(1)}|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle); \sqrt{\mathtt{I}}^{(1)}|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

Hence, $\sqrt{\mathtt{I}}^{(1)}$ transforms bits into genuine qubits.

Definition 5 *(The square root of* NOT*)* For any $n \geq 1$, the *square root of* NOT (defined on $\mathcal{H}^{(n)}$) is the linear operator $\sqrt{\mathtt{NOT}}^{(n)}$ such that for every element $|x_1, \ldots, x_n\rangle$ of the canonical basis:

$$\sqrt{\mathtt{NOT}}^{(n)}|x_1, \ldots, x_n\rangle = |x_1, \ldots, x_{n-1}\rangle \otimes \left(\frac{1-i}{2}|x_n\rangle + \frac{1+i}{2}|1 - x_n\rangle\right),$$

where $i = \sqrt{-1}$.

Definition 6 (*The Fredkin-gate*) For any $m, n, p \geq 1$, the Fredkin-gate (defined on $\mathcal{H}^{(m+n+p)}$) is the linear operator $\mathtt{F}^{(m,n,p)}$ such that for every element $|x_1, \ldots, x_m, y_1, \ldots, y_n, z_1, \ldots, z_p\rangle$ of the canonical basis of $\mathcal{H}^{(m+n+p)}$,

$$\mathtt{F}^{(m,n,p)}|x_1, \ldots, x_m, y_1, \ldots, y_n, z_1, \ldots, z_p\rangle = \begin{cases} |x_1, \ldots, x_{m-1}, y_n, y_1, \ldots, y_{n-1}, x_m, z_1, \ldots, z_p\rangle, & \text{if } z_p = 0; \\ |x_1, \ldots, x_m, y_1, \ldots, y_n, z_1, \ldots, z_p\rangle, & \text{otherwise.} \end{cases}$$

Definition 7 *(The shift-gate)* For any $n \geq 1$, the *shift-gate* (defined on $\mathcal{H}^{(n)}$) is the linear operator $\mathtt{SH}^{(n)}$ such that, for every element $|x_1 \ldots, x_n\rangle$ of the canonical basis,

$$\mathtt{SH}^{(n)}|x_1, \ldots, x_n\rangle = |x_n, x_1, \ldots, x_{n-1}\rangle.$$

Now we introduce a highly representative class of gates, called *proper unitary connective-gates* (briefly, *connective-gates*).

Definition 8 (*Connective-gate*) Consider a Hilbert space $\mathcal{H}^{(m+n+p)}$ with $m, n \geq 0$ and $p \geq 1$.

A *connective-gate* of $\mathcal{H}^{(m+n+p)}$ is a unitary operator $\mathtt{G}^{(m,n,p)}$ that can be represented in the following form:

$$\mathtt{G}^{(m,n,p)} := \left[P_0^{(m)} \otimes P_0^{(n)} \otimes \mathtt{I}^{(p-1)} \otimes U_{00}\right] + \left[P_0^{(m)} \otimes P_1^{(n)} \otimes \mathtt{I}^{(p-1)} \otimes U_{01}\right] + \left[P_1^{(m)} \otimes P_0^{(n)} \otimes \mathtt{I}^{(p-1)} \otimes U_{10}\right] + \left[P_1^{(m)} \otimes P_1^{(n)} \otimes \mathtt{I}^{(p-1)} \otimes U_{11}\right],$$

where U_{ij} are unitary operators of $\mathcal{H}^{(1)}$, $\mathtt{I}^{(0)} = 1$, $P_0^{(0)} = P_1^{(0)} = \frac{1}{2}$.[1]

Apparently, any connective-gate $\mathtt{G}^{(n)}$ applied to a register $|x_1, \ldots, x_n\rangle$ transforms the target-bit $|x_n\rangle$ into a qubit that determines the probability-value of $\mathtt{G}^{(n)}(|x_1, \ldots, x_n\rangle)$.

One can easily check that the negation $\mathtt{NOT}^{(1)}$ (of $\mathcal{H}^{(1)}$) and the Toffoli-gate $\mathtt{T}^{(1,1,1)}$ (of $\mathcal{H}^{(1+1+1)}$) are connective-gates, since they can be represented in the following form:

$$\mathtt{NOT}^{(1)} = \left(\frac{1}{2} \cdot \frac{1}{2} \cdot 1\right)\mathtt{NOT}^{(1)} + \left(\frac{1}{2} \cdot \frac{1}{2} \cdot 1\right)\mathtt{NOT}^{(1)} + \left(\frac{1}{2} \cdot \frac{1}{2} \cdot 1\right)\mathtt{NOT}^{(1)} + \left(\frac{1}{2} \cdot \frac{1}{2} \cdot 1\right)\mathtt{NOT}^{(1)};$$

$$\mathtt{T}^{(1,1,1)} = \left[P_0^{(1)} \otimes P_0^{(1)} \otimes 1 \cdot \mathtt{I}^{(1)}\right] + \left[P_0^{(1)} \otimes P_1^{(1)} \otimes 1 \cdot \mathtt{I}^{(1)}\right] + \left[P_1^{(1)} \otimes P_0^{(1)} \otimes 1 \cdot \mathtt{I}^{(1)}\right] + \left[P_1^{(1)} \otimes P_1^{(1)} \otimes 1 \cdot \mathtt{NOT}^{(1)}\right].$$

Similar representations can be given for the gates $\mathtt{NOT}^{(p)}, \mathtt{T}^{(m,n,p)}, \mathtt{XOR}^{(n,p)}$, $\sqrt{\mathtt{NOT}}^{(p)}$, $\sqrt{\mathtt{I}}^{(p)}$. Examples of gates that are not connective-gates are the Fredkin-gate and the shift-gate.

[1] We recall that $P_0^{(n)}$ and $P_1^{(n)}$ represent, respectively, the falsity-property and the truth-property of the space $\mathcal{H}^{(n)}$ (see Section 2 of the first part of this article).

All gates can be canonically extended to the set $\mathfrak{D}$ of all density operators. Let $\mathtt{G}$ be any gate defined on $\mathcal{H}^{(n)}$. The corresponding *density-operator gate* (also called *unitary quantum operation*) ${}^{\mathfrak{D}}\mathtt{G}$ is defined as follows for any $\rho \in \mathfrak{D}(\mathcal{H}^{(n)})$:

$$ {}^{\mathfrak{D}}\mathtt{G}\rho = \mathtt{G}\,\rho\,\mathtt{G}^{\dagger} \;\text{ where }\; \mathtt{G}^{\dagger} \;\text{ is the adjoint of }\; \mathtt{G}. $$

For the sake simplicity, also the operations ${}^{\mathfrak{D}}\mathtt{G}$ will be briefly called *gates*.

Theorem 1 *There is no connective-gate $\mathtt{G}^{(m,n,1)}$ that satisfies the following condition: there exists a function $\mu : [0,1] \to [0,1]$ such that for any density operator ρ of $\mathcal{H}^{(m)} \otimes \mathcal{H}^{(n)}$, $\mu\left(\mathtt{p}\left({}^{\mathfrak{D}}\mathtt{G}^{(m,n,1)}\left(\rho \otimes P_0^{(1)}\right)\right)\right)$ is an entanglement-measure for ρ (where* p *is the probability-function defined on the set of all density operators).*[2]

Proof Suppose there exists a connective gate $\mathtt{G}^{(m,n,1)}$ satisfying the condition considered by the theorem.
Take $\rho_1 = P_{\frac{1}{\sqrt{2}}(|0,\ldots,0,0\rangle + |1,\ldots,1,0\rangle)}$ and $\rho_2 = \frac{1}{2}\left(P_{|0,\ldots,0,0\rangle} + P_{|1,\ldots,1,0\rangle}\right)$. Clearly, ρ_1 is entangled, while ρ_2 is separable. At the same time we have: $\mathtt{p}({}^{\mathfrak{D}}\mathtt{G}^{(m,n,1)}(\rho_1)) = \mathrm{tr}\left(P_1^{(m+n+1)} P_{\frac{1}{\sqrt{2}}(|0,\ldots,0\rangle\otimes U_{00}|0\rangle + |1,\ldots,1\rangle\otimes U_{11}|0\rangle)}\right) = \mathrm{tr}\left(P_1^{(1)}\,{}^{\mathfrak{D}}U_{00}\left(P_0^{(1)}\right)\right) + \mathrm{tr}\left(P_1^{(1)}\,{}^{\mathfrak{D}}U_{11}\left(P_0^{(1)}\right)\right) = \mathrm{tr}\left(P_1^{(m+n+1)}\frac{1}{2}(P_{|0,\ldots,0\rangle\otimes U_{00}|0\rangle} + P_{|1,\ldots,1\rangle\otimes U_{11}|0\rangle})\right) = \mathtt{p}({}^{\mathfrak{D}}\mathtt{G}^{(m,n,1)}(\rho_2))$. □

This theorem implies that the probabilistic behavior of connective-gates cannot characterize entanglement.

2 Truth-perspectives and Gates

As is well known, the choice of an orthonormal basis for the space $\mathbb{C}^2$ is a matter of convention. One can consider infinitely many bases that are determined by the application of a unitary operator $\mathfrak{T}$ to the elements of the canonical basis ($|1\rangle$ and $|0\rangle$). From an intuitive point of view, we can think that the operator $\mathfrak{T}$ gives rise to a change of *truth-perspective*. Different epistemic agents $\mathfrak{a}$ can be associated to different truth-perspectives $\mathfrak{T}_{\mathfrak{a}}$, because their knowledge is based on different ideas about *truth* and *falsity*.[3] From a physical point of view, we can suppose that each truth-perspective corresponds to an apparatus that allows one to measure a given observable.

Any truth-perspective $\mathfrak{T}$ can be naturally extended to a unitary operator $\mathfrak{T}^{(n)}$ of $\mathcal{H}^{(n)}$ (for any $n \geq 1$):

$$ \mathfrak{T}^{(n)}|x_1,\ldots,x_n\rangle = \mathfrak{T}|x_1\rangle \otimes \ldots \otimes \mathfrak{T}|x_n\rangle. $$

Accordingly, any choice of $\mathfrak{T}$ determines an orthonormal basis $B_{\mathfrak{T}}^{(n)}$ for $\mathcal{H}^{(n)}$ such that:

$$ B_{\mathfrak{T}}^{(n)} = \left\{\mathfrak{T}^{(n)}|x_1,\ldots,x_n\rangle : |x_1,\ldots,x_n\rangle \in B_{\mathtt{I}}^{(n)}\right\}, $$

[2] We recall that $\mathtt{p}(\rho) := \mathrm{tr}(P_1^{(n)}\,\rho)$, for any density operator ρ of $\mathcal{H}^{(n)}$, where tr is the trace-functional (see Section 2 of the first part of this article).

[3] Truth-perspectives play an important role in the case of *epistemic quantum computational logics*. See, for instance, [1–3].

where $B_{\mathtt{I}}^{(n)}$ is the canonical basis of $\mathcal{H}^{(n)}$.

Instead of $\mathfrak{T}^{(n)}|x_1, \ldots, x_n\rangle$ we will also write: $|x_{1_{\mathfrak{T}}}, \ldots, x_{n_{\mathfrak{T}}}\rangle$. The elements of $B_{\mathfrak{T}}^{(n)}$ represent the $\mathfrak{T}$-*registers* of $\mathcal{H}^{(n)}$, while $|1_{\mathfrak{T}}\rangle$ and $|0_{\mathfrak{T}}\rangle$ represent the truth-values *Truth* and *Falsity* with respect to the truth-perspective $\mathfrak{T}$.

The notions of *truth*, *falsity* and *probability* with respect to a truth - perspective $\mathfrak{T}$ can be defined like in the canonical case. The $\mathfrak{T}$-truth ${}^{\mathfrak{T}}P_1^{(n)}$ is the projection-operator that projects over the closed subspace spanned by the set of all $\mathfrak{T}$-registers $|x_{1_{\mathfrak{T}}}, \ldots, x_{(n-1)_{\mathfrak{T}}}, 1_{\mathfrak{T}}\rangle$; while the $\mathfrak{T}$-falsity ${}^{\mathfrak{T}}P_0^{(n)}$ is the projection-operator that projects over the closed subspace spanned by the set of all $\mathfrak{T}$-registers $|x_{1_{\mathfrak{T}}}, \ldots, x_{(n-1)_{\mathfrak{T}}}, 0_{\mathfrak{T}}\rangle$. The $\mathfrak{T}$-probability $\mathtt{p}_{\mathfrak{T}}$ is defined as follows for any density operator ρ of $\mathcal{H}^{(n)}$:

$$\mathtt{p}_{\mathfrak{T}}(\rho) := \mathrm{tr}({}^{\mathfrak{T}}P_1^{(n)}\rho).$$

All gates can be naturally transposed from the canonical truth-perspective to any truth-perspective $\mathfrak{T}$. Let $\mathtt{G}^{(n)}$ be any gate defined with respect to the canonical truth-perspective. The *twin-gate* $\mathtt{G}_{\mathfrak{T}}^{(n)}$, defined with respect to the truth-perspective $\mathfrak{T}$, is determined as follows:

$$\mathtt{G}_{\mathfrak{T}}^{(n)} := \mathfrak{T}^{(n)}\mathtt{G}^{(n)}\mathfrak{T}^{(n)^\dagger}.$$

In the following we will almost always refer to the canonical truth-perspective and to canonical gates.

3 Anti-unitary Gates and Pseudo-unitary Gates

To what extent are we forced to assume that quantum information should be always processed by quantum unitary operations? Are there any alternative reasonable choices?

An interesting discussion has concerned the role of negation in quantum computation. As is well known, the gate $\mathtt{NOT}^{(n)}$ has a characteristic property that seems to be counter-intuitive: a quregister $|\psi\rangle$ (of $\mathcal{H}^{(n)}$) is not necessarily orthogonal to its negation $\mathtt{NOT}^{(n)}|\psi\rangle$. For instance, the qubit $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ is a fixed point of $\mathtt{NOT}^{(1)}$:

$$\mathtt{NOT}^{(1)}\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle).$$

At the same time, for all registers $|x_1, \ldots, x_n\rangle$ we have:

$$|x_1, \ldots, x_n\rangle \perp \mathtt{NOT}^{(n)}|x_1, \ldots, x_n\rangle.$$

One can prove that no unitary operator U of $\mathcal{H}^{(1)}$ satisfies, at the same time, the two following conditions (which seem both desirable properties for a gate representing a faithful generalization of the classical negation):

1. $|\psi\rangle \perp U|\psi\rangle$, for any qubit $|\psi\rangle$;
2. $\mathtt{p}({}^{\mathfrak{D}}(UU)P_{|\psi\rangle}) = \mathtt{p}(P_{|\psi\rangle})$, for any qubit $|\psi\rangle$ (where $\mathtt{p}$ is the probability-function defined with respect to the canonical truth-perspective).

This intuitive shortcoming can be overcome, if we decide to replace the familiar negation $\mathtt{NOT}^{(n)}$ (which is a unitary operator) with a particular example of an *anti-unitary operator*.

Let us first recall the general definition of *anti-unitary operator*.

Definition 9 (*Anti-unitary operator*) An operator A of a Hilbert space $\mathcal{H}$ is called *anti-unitary* iff A satisfies the following conditions:

1) A preserves the absolute value of inner product.

2) A is anti-linear. In other words, for any vector $\sum_i c_i|\psi_i\rangle$ of $\mathcal{H}$:

$$A\sum_i c_i|\psi_i\rangle = \sum_i c_i^* A|\psi_i\rangle.$$

As is well known, anti-unitary operators play an important role in quantum field theory [8], where they are applied, for instance, to represent time-reversal symmetry. In quantum computation these operators have been recently used for different aimes (for instance, in all criteria of inseparability for two-qubit systems and in the quantum simulation of the Majorana-equation)[4].

The *anti-unitary negation* (also called *universal* NOT) is defined as follows.

Definition 10 (*The anti-unitary negation*) For any $n \geq 1$, the *anti-unitary negation* is the anti-linear operator $\widehat{\mathtt{NOT}}^{(n)}$ defined on $\mathcal{H}^{(n)}$ such that for every element $|x_1, \ldots, x_n\rangle$ of the canonical basis,

$$\widehat{\mathtt{NOT}}^{(n)}|x_1, \ldots, x_n\rangle = (-1)^{1-x_n}|x_1, \ldots, x_{n-1}, 1 - x_n\rangle.$$

One can easily show that:

1. $|\psi\rangle \perp \widehat{\mathtt{NOT}}^{(n)}|\psi\rangle$, for any $|\psi\rangle$ of $\mathcal{H}^{(n)}$;
2. $\widehat{\mathtt{NOT}}^{(n)}\widehat{\mathtt{NOT}}^{(n)}|\psi\rangle = -|\psi\rangle$, for any $|\psi\rangle$ of $\mathcal{H}^{(n)}$; hence, $\mathtt{p}\left({}^{\mathfrak{D}}\left(\widehat{\mathtt{NOT}}^{(n)}\widehat{\mathtt{NOT}}^{(n)}\right)P_{|\psi\rangle}\right) = \mathtt{p}(P_{|\psi\rangle})$.

The anti-unitary negation admits an intuitive geometrical interpretation in the case of the space $\mathbb{C}^2$. Consider the Poincaré-Bloch sphere, which is in one-to-one correspondence with the set $\mathfrak{D}(\mathbb{C}^2)$ of all density operators of $\mathbb{C}^2$. In such a case, $\widehat{\mathtt{NOT}}^{(1)}$ transforms any point of the sphere into its antipodal point.

It may be interesting to consider a proper set of $\mathfrak{D}(\mathbb{C}^2)$, corresponding to the part of the sphere that is represented by the dashed circle (in Fig. 1).

Let $\mathfrak{T}$ be the truth-perspective defined as follows: $\mathfrak{T}|0\rangle = |0\rangle$; $\mathfrak{T}|1\rangle = i|1\rangle$; and consider the gate ${}^{\mathfrak{D}}\mathtt{NOT}_{\mathfrak{T}}^{(1)}$ (the twin-gate of ${}^{\mathfrak{D}}\mathtt{NOT}^{(1)}$ determined by $\mathfrak{T}$). We have: ${}^{\mathfrak{D}}\mathtt{NOT}_{\mathfrak{T}}^{(1)} = {}^{\mathfrak{D}}\sigma_y$, where σ_y is the second Pauli matrix. Hence, ${}^{\mathfrak{D}}\mathtt{NOT}_{\mathfrak{T}}^{(1)}$ transforms any point of the dashed circle into its antipodal point; consequently: ${}^{\mathfrak{D}}\mathtt{NOT}_{\mathfrak{T}}^{(1)}\rho = {}^{\mathfrak{D}}\widehat{\mathtt{NOT}}^{(1)}\rho$, for any ρ corresponding to a point of our circle. This justifies the following conclusion: for some pieces of quantum information (corresponding to points of the dashed circle) and for some epistemic agents (whose truth-perspective is $\mathfrak{T}$) the standard unitary negation and the anti-unitary negation coincide.

An interesting investigation concerns possible experimental realizations of approximations of some anti-unitary gates via unitary gates. An approximation of the gate $\widehat{\mathtt{NOT}}^{(1)}$ has been obtained by using a slight modification of the standard quantum teleportation-protocol [9]. Other techniques allow us to approximate $\widehat{\mathtt{NOT}}^{(n)}$ (as well as other anti-unitary gates) by means of completely positive maps [7].

[4] See [4, 6].

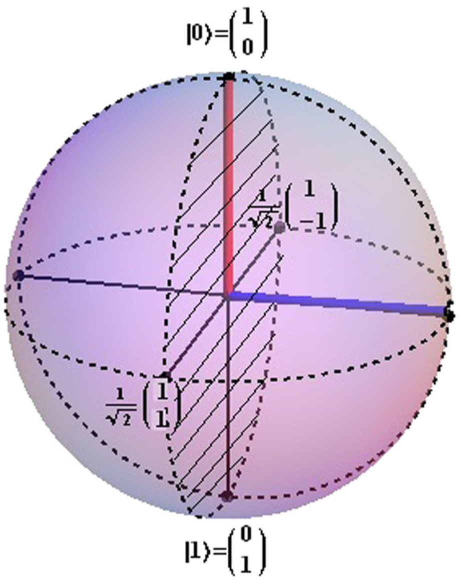

Fig. 1 The Poincaré-Bloch sphere

Have all connective-gates a natural anti-unitary counterpart? The answer to this question is negative. Consider the case of the square-root of the anti-unitary negation. One can prove that any operator $\widehat{\sqrt{\mathtt{NOT}}}^{(n)}$ such that

$$\widehat{\sqrt{\mathtt{NOT}}}^{(n)}\widehat{\sqrt{\mathtt{NOT}}}^{(n)} = \widehat{\mathtt{NOT}}^{(n)}$$

is neither linear nor anti-linear. This suggests to consider a larger class of operators that we call *pseudo-unitary gates*[11].

Definition 11 (*Pseudo-unitary gate*) The set of all pseudo-unitary gates (also called *additive bounded operators*) of a Hilbert space $\mathcal{H}$ is the smallest set $Ps(\mathcal{H})$ such that

1) $Ps(\mathcal{H})$ is included in the set of all bounded operators of $\mathcal{H}$;
2) $Ps(\mathcal{H})$ includes the set of all unitary operators and the set of all anti-unitary operators of $\mathcal{H}$;
3) $Ps(\mathcal{H})$ is closed under operator-sum.

Of course, $\widehat{\mathtt{NOT}}^{(n)}$ is (trivially) an example of a pseudo-unitary operator. The pseudo-unitary square-root of negation, the pseudo-unitary square-root of identity, the pseudo-unitary Toffoli-gate and the pseudo-unitary $\widehat{\mathtt{XOR}}$-gate can be defined as follows.

Definition 12 (*The pseudo-unitary square-root of negation*) For any $n \geq 1$, the pseudo-unitary square-root of negation on $\mathcal{H}^{(n)}$ is defined as follows:

$$\widehat{\sqrt{\mathtt{NOT}}}^{(n)} = \frac{1}{\sqrt{2}}(\mathtt{I}^{(n)} + \widehat{\mathtt{NOT}}^{(n)}).$$

We have:

1. $\widehat{\sqrt{\mathtt{NOT}}}^{(n)}$ is (by definition) a pseudo-unitary gate;
2. $\widehat{\sqrt{\mathtt{NOT}}}^{(n)}\widehat{\sqrt{\mathtt{NOT}}}^{(n)} = \widehat{\mathtt{NOT}}^{(n)}$.

Definition 13 (*The pseudo-unitary square-root of identity*) For any $n \geq 1$, the pseudo-unitary square-root of identity on $\mathcal{H}^{(n)}$ is defined as follows:

$$\widehat{\sqrt{\mathtt{I}}}^{(n)} = \frac{1}{\sqrt{2}}(\mathtt{I}^{(n)} - \widehat{\mathtt{NOT}}^{(n)})(\mathtt{I}^{(n-1)} \otimes \sigma_z),$$

where σ_z is the third Pauli-matrix.

We have:

1. $\widehat{\sqrt{\mathtt{I}}}^{(n)}$ is (by definition) a pseudo-unitary gate:
2. $\widehat{\sqrt{\mathtt{I}}}^{(n)}\widehat{\sqrt{\mathtt{I}}}^{(n)} = \mathtt{I}^{(n)}$.

Definition 14 (*The pseudo-unitary Toffoli-gate*) For any $m, n, p \geq 1$, the pseudo-unitary Toffoli-gate on $\mathcal{H}^{(m+n+p)}$ is defined as follows:

$$\widehat{\mathtt{T}}^{(m,n,p)} = \left(\mathtt{I}^{(m+n)} - P_1^{(m)} \otimes P_1^{(n)}\right) \otimes \mathtt{I}^{(p)} + P_1^{(m)} \otimes P_1^{(n)} \otimes \widehat{\mathtt{NOT}}^{(p)}.$$

Definition 15 (*The pseudo-unitary* XOR-*gate*) For any $m, n \geq 1$, the pseudo-unitary XOR-gate on $\mathcal{H}^{(m+n)}$ is defined as follows:

$$\widehat{\mathtt{XOR}}^{(m,n)} = P_0^{(m)} \otimes \mathtt{I}^{(n)} + P_1^{(m)} \otimes \widehat{\mathtt{NOT}}^{(n)}.$$

As happens in the case of unitary gates, also pseudo-unitary gates can be generalized to density operators. If $\widehat{\mathtt{G}}^{(n)}$ is a pseudo-unitary operator of $\mathcal{H}^{(n)}$, the corresponding *pseudo-unitary quantum operation* ${}^{\mathfrak{D}}\widehat{\mathtt{G}}^{(n)}$ is defined on the set of all density operators ρ of $\mathcal{H}^{(n)}$ as follows:

$${}^{\mathfrak{D}}\widehat{\mathtt{G}}^{(n)}\rho = \widehat{\mathtt{G}}^{(n)}\rho\,\widehat{\mathtt{G}}^{(n)\dagger}.$$

Both the pseudo-unitary operators $\widehat{\mathtt{G}}^{(n)}$ and the pseudo-unitary quantum operations ${}^{\mathfrak{D}}\widehat{\mathtt{G}}^{(n)}$ will be called *pseudo-unitary gates*.

As expected, a unitary gate ${}^{\mathfrak{D}}\mathtt{G}^{(n)}$ and its correspondent pseudo-unitary gate ${}^{\mathfrak{D}}\widehat{\mathtt{G}}^{(n)}$ generally give rise to different outputs, when applied to one and the same input:

$${}^{\mathfrak{D}}\mathtt{G}^{(n)}\rho \neq {}^{\mathfrak{D}}\widehat{\mathtt{G}}^{(n)}\rho.$$

In spite of this, one can prove that applications of the gates negation, Toffoli, XOR, square-root of negation, square-root of identity determine the same probability-values in the unitary and in the pseudo-unitary case.

Theorem 2 1. *For any ρ of $\mathcal{H}^{(n)}$,*

$$\mathtt{p}\left({}^{\mathfrak{D}}\widehat{\mathtt{NOT}}^{(n)}\rho\right) = \mathtt{p}\left({}^{\mathfrak{D}}\mathtt{NOT}^{(n)}\rho\right),$$
$$\mathtt{p}\left({}^{\mathfrak{D}}\widehat{\sqrt{\mathtt{NOT}}}^{(n)}\rho\right) = \mathtt{p}\left({}^{\mathfrak{D}}\sqrt{\mathtt{NOT}}^{(n)}\rho\right),$$
$$\mathtt{p}\left({}^{\mathfrak{D}}\widehat{\sqrt{\mathtt{I}}}^{(n)}\rho\right) = \mathtt{p}\left({}^{\mathfrak{D}}\sqrt{\mathtt{I}}^{(n)}\rho\right).$$

2. *For any* ρ *of* $\mathcal{H}^{(m+n+p)}$,
$\mathtt{p}\left({}^{\mathfrak{D}}\widehat{\mathtt{T}}^{(m,n,p)}\rho\right) = \mathtt{p}({}^{\mathfrak{D}}\mathtt{T}^{(m,n,p)}\rho)$.
3. *For any* ρ *of* $\mathcal{H}^{(m+n)}$,
$\mathtt{p}\left({}^{\mathfrak{D}}\widehat{\mathtt{XOR}}^{(m,n)}\rho\right) = \mathtt{p}\left({}^{\mathfrak{D}}\mathtt{XOR}^{(m,n)}\rho\right)$.

Proof

$$
\begin{aligned}
&1.\mathrm{p}\left({}^{\mathcal{D}}\widehat{\mathtt{NOT}}^{(n)}\rho\right)\\
&= \mathrm{tr}\left(P_1^{(n)}\widehat{\mathtt{NOT}}^{(n)}\rho\,\widehat{\mathtt{NOT}}^{(n)\dagger}\right)\\
&= \mathrm{tr}\left(\widehat{\mathtt{NOT}}^{(n)\dagger}P_1^{(n)}\widehat{\mathtt{NOT}}^{(n)}\rho\right)\\
&= \mathrm{tr}\left(\left(\mathtt{I}^{(n-1)}\otimes\widehat{\mathtt{NOT}}^{(1)\dagger}\right)\left(\mathtt{I}^{(n-1)}\otimes P_1^{(1)}\right)\left(\mathtt{I}^{(n-1)}\otimes\widehat{\mathtt{NOT}}^{(1)}\right)\rho\right)\\
&= \mathrm{tr}\left(\left(\mathtt{I}^{(n-1)}\otimes\widehat{\mathtt{NOT}}^{(1)\dagger}P_1^{(1)}\widehat{\mathtt{NOT}}^{(1)}\right)\rho\right)\\
&= \mathrm{tr}\left(\left(\mathtt{I}^{(n-1)}\otimes\mathtt{NOT}^{(1)\dagger}P_1^{(1)}\mathtt{NOT}^{(1)}\right)\rho\right)\\
&= \mathrm{tr}\left(\mathtt{NOT}^{(n)\dagger}P_1^{(n)}\mathtt{NOT}^{(n)}\rho\right)\\
&= \mathrm{tr}\left(P_0^{(n)}\rho\right)\\
&= \mathrm{tr}\left(\left(I^{(n)}-P_1^{(n)}\right)\rho\right)\\
&= 1-\mathrm{p}(\rho).
\end{aligned}
$$

In a similar way for $\widehat{\sqrt{\mathtt{NOT}}}^{(n)}$ and $\widehat{\sqrt{\mathtt{I}}}^{(n)}$.

$$
\begin{aligned}
&2.\mathrm{p}({}^{\mathcal{D}}\widehat{\mathtt{T}}^{(m,n,p)}\rho)\\
&= \mathrm{tr}(P_1^{(m+n+p)}\widehat{\mathtt{T}}^{(m,n,p)}\rho\,\widehat{\mathtt{T}}^{(m,n,p)\dagger})\\
&= \mathrm{tr}(((\mathtt{I}^{(m+n)}-\mathtt{I}^{(m-1)}\otimes P_1^{(1)}\otimes\mathtt{I}^{(n-1)}\otimes P_1^{(1)})\otimes\mathtt{I}^{(p)}\\
&+\mathtt{I}^{(m-1)}\otimes P_1^{(1)}\otimes\mathtt{I}^{(n-1)}\otimes P_1^{(1)}\otimes\mathtt{I}^{(p-1)}\otimes\widehat{\mathtt{NOT}}^{(1)})^\dagger\\
&(\mathtt{I}^{(m+n+p-1)}\otimes P_1^{(1)})\\
&((\mathtt{I}^{(m+n)}-\mathtt{I}^{(m-1)}\otimes P_1^{(1)}\otimes\mathtt{I}^{(n-1)}\otimes P_1^{(1)})\otimes\mathtt{I}^{(p)}\\
&+\mathtt{I}^{(m-1)}\otimes P_1^{(1)}\otimes\mathtt{I}^{(n-1)}\otimes P_1^{(1)}\otimes\mathtt{I}^{(p-1)}\otimes\widehat{\mathtt{NOT}}^{(1)})\rho)\\
&= \mathrm{p}({}^{\mathcal{D}}\mathtt{T}^{(m,n,p)}\rho).
\end{aligned}
$$

3. In a similar way for $\mathtt{XOR}^{(m,n)}$. □

In the next Section we will see how this theorem has an important consequence for the development of a *pseudo-unitary* version of the standard (*unitary*) holistic quantum computational semantics.

4 Unitary and Pseudo-unitary Quantum Computational Semantics

In the standard semantics for quantum computational logics formulas are interpreted as pieces of quantum information (density operators of convenient spaces $\mathcal{H}^{(n)}$), while the logical connectives are interpreted as unitary gates.[5] This approach can be naturally transformed into a *pseudo-unitary semantics*, where the same connectives are interpreted as pseudo-unitary gates. Let us briefly recall the basic concepts of the unitary semantics. The alphabet of the quantum computational language $\mathcal{L}$ contains atomic formulas ($\mathbf{q}, \mathbf{q}_1, \mathbf{q}_2, \ldots$) including two privileged formulas $\mathbf{t}$ and $\mathbf{f}$ that represent the truth-values *Truth* and *Falsity* respectively. The connectives of $\mathcal{L}$ are: the negation $\neg$ (corresponding to the gate $\mathtt{NOT}^{(n)}$), the ternary Toffoli-connective $\intercal$ (corresponding to the gate $\mathtt{T}^{(m,n,p)}$), the exclusive disjunction $\uplus$ (corresponding to the gate $\mathtt{XOR}^{(m,n)}$), the square root of negation $\sqrt{\neg}$ (corresponding to the gate $\sqrt{\mathtt{NOT}}^{(n)}$), the square root of identity $\sqrt{id}$ (corresponding to the gate $\sqrt{\mathtt{I}}^{(n)}$). The notion of *formula* is inductively defined as follows: 1) atomic formulas are formulas; 2) if $\alpha,\ \beta,\ \gamma$ are formulas, then $\neg\alpha$, $\sqrt{\neg}\alpha$, $\sqrt{id}\alpha$, $\intercal(\alpha, \beta, \gamma)$, $\alpha \uplus \beta$ are formulas. Recalling the definition of the holistic conjunction $\mathtt{AND}^{(m,n)}$, a binary conjunction $\wedge$ can be defined in terms of the Toffoli-connective: $\alpha \wedge \beta := \intercal(\alpha, \beta, \mathbf{f})$ (where $\mathbf{f}$ plays the role of a *syntactical ancilla*)[6].

By *atomic complexity* of a formula α we mean the number $At(\alpha)$ of occurrences of atomic subformulas in α. For instance, the atomic complexity of the formula $\alpha = \mathbf{q} \wedge \neg\mathbf{q} = \intercal(\mathbf{q}, \neg\mathbf{q}, \mathbf{f})$ is 3. The number $At(\alpha)$ plays an important semantic role, since it determines the *semantic space* $\mathcal{H}^{\alpha} = \mathcal{H}^{(At(\alpha))}$, where any density operator representing a possible *informational meaning* of α shall live. We have, for instance, $\mathcal{H}^{\intercal(\mathbf{q},\neg\mathbf{q},\mathbf{f})} = \mathcal{H}^{(3)}$.

Any formula α can be naturally decomposed into its parts giving rise to a special configuration, called the *syntactical tree* of α ($STree^{\alpha}$). Roughly, $STree^{\alpha}$ can be represented as a sequence of *levels* consisting of subformulas of α. The *bottom-level* is (α), while all other levels are obtained by dropping, step by step, all connectives occurring in α. Hence, the *top-level* is the sequence of atomic formulas occurring in α. As an example consider again the formula $\alpha = \intercal(\mathbf{q}, \neg\mathbf{q}, \mathbf{f})$. In such a case, $STree^{\alpha}$ is the following sequence of *levels*:

$$\begin{aligned} Level_3^{\alpha} &= (\mathbf{q}, \mathbf{q}, \mathbf{f}) \\ Level_2^{\alpha} &= (\intercal(\mathbf{q}, \neg\mathbf{q}, \mathbf{f})) \\ Level_1^{\alpha} &= (\intercal(\mathbf{q}, \neg\mathbf{q}, \mathbf{f})) \end{aligned}$$

For any α, $STree^{\alpha}$ uniquely determines the *gate-tree* of α: a sequence of gates all defined on the space $\mathcal{H}^{\alpha}$. As an example, consider again the formula, $\alpha = \intercal(\mathbf{q}, \neg\mathbf{q}, \mathbf{f})$. In the syntactical tree of α the second level has been obtained (from the third level) by repeating the first occurrence of $\mathbf{q}$, by negating the second occurrence of $\mathbf{q}$ and by repeating $\mathbf{f}$; while the first level has been obtained (from the second level) by applying the Toffoli-connective. Accordingly, the gate-tree of α can be naturally identified with the following gate-sequence:

$$\left({}^{\mathfrak{D}}(\mathtt{I}^{(1)} \otimes \mathtt{NOT}^{(1)} \otimes \mathtt{I}^{(1)})^{\mathfrak{D}}\mathtt{T}^{(1,1,1)}\right).$$

[5]See [5].

[6]We recall that for any $m, n \geq 1$ and for any density operator ρ of the Hilbert space $\mathcal{H}^{(m+n)}$, $\mathtt{AND}^{(m,n)}$ is defined as follows: $\mathtt{AND}^{(m,n)}(\rho) := {}^{\mathfrak{D}}\mathtt{T}^{(m,n,1)}(\rho \otimes P_0^{(1)})$ (see Definition 7 of the first part of this article).

This procedure can be naturally generalized to any α, whose gate-tree will be indicated by $\left({}^{\mathfrak{D}}\mathtt{G}^{\alpha}_{n-1}, \ldots, {}^{\mathfrak{D}}\mathtt{G}^{\alpha}_{1}\right)$ (where n is the number of levels of $STree^{\alpha}$). On this basis, we can say that any formula of $\mathcal{L}$ can be regarded as a synthetic logical description of a quantum circuit.

We consider here a *holistic* version of the quantum computational semantics, where entanglement can be used as a "semantic resource"[5]. Generally, the meaning of a compound formula determines the *contextual* meanings of its parts (and not the other way around, as happens in the case of most *compositional* semantic approaches).

The concept of *model* of $\mathcal{L}$ is based on the notion of *holistic map* for $\mathcal{L}$. This is a map $\mathtt{Hol}$ such that, for any α and for each $Level^{\alpha}_{i}$ (of $STree^{\alpha}$),

$$\mathtt{Hol} : \alpha \;\; \mapsto \;\; \rho \in \mathfrak{D}(\mathcal{H}^{\alpha}).$$

Suppose that $Level^{\alpha}_{i} = (\beta_{i_1}, \ldots, \beta_{i_r})$. It is natural to describe $\rho = \mathtt{Hol}(Level^{\alpha}_{i})$ as a possible state of a composite quantum system consisting of r subsystems. Hence, the *contextual meaning* $(\mathtt{Hol}^{\alpha}(\beta_{i_j}))$ of the occurrence β_{i_j} (in $STree^{\alpha}$) can be identified with the reduced state of ρ with respect to the j-th subsystem. Accordingly, we can write:

$$\mathtt{Hol}^{\alpha}(\beta_{i_j}) = Red^{(j)}_{[At(\beta_{i_1}),\ldots,At(\beta_{i_r})]}(\rho).$$

In a similar way one can also define the contextual meaning of a subsequence $(\beta_{i_{k_1}}, \ldots, \beta_{i_{k_u}})$ of $Level^{\alpha}_{i}$.

The concepts of *model*, *truth* and *logical consequence* are now defined as follows.

Definition 16 (*Model*) A *model* of the language $\mathcal{L}$ is a holistic map $\mathtt{Hol}$ that satisfies the following conditions for any formula α:

1) $\mathtt{Hol}$ assigns the same contextual meaning to different occurrences of one and the same subformula of α (in $STree^{\alpha}$).
2) The contextual meanings of the true formula **t** and of the false formula **f** are the truth $P^{(1)}_{1}$ and the falsity $P^{(1)}_{0}$, respectively.
3) $\mathtt{Hol}$ preserves the logical form of α by interpreting the connectives of α as the corresponding gates. Accordingly, if $\left({}^{\mathfrak{D}}\mathtt{G}^{\alpha}_{n-1}, \ldots, {}^{\mathfrak{D}}\mathtt{G}^{\alpha}_{1}\right)$ is the gate-tree of α, then $\mathtt{Hol}(Level^{\alpha}_{i}) = {}^{\mathfrak{D}}\mathtt{G}^{\alpha}_{i}\left(\mathtt{Hol}\left(Level^{\alpha}_{i+1}\right)\right)$.

On this basis we put:

$$\mathtt{Hol}(\alpha) = \mathtt{Hol}(Level^{\alpha}_{1}), \quad \text{for any formula } \alpha.$$

Notice that any $\mathtt{Hol}(\alpha)$ represents a kind of autonomous semantic context that is not necessarily correlated with the meanings of other formulas. Generally we have: $\mathtt{Hol}^{\gamma}(\beta) \neq \mathtt{Hol}^{\delta}(\beta)$. Thus, one and the same formula may receive different contextual meanings in different contexts.

Definition 17 (*Truth*) A formula α is called *true* with respect to a model $\mathtt{Hol}$ iff $\mathtt{p}(\mathtt{Hol}(\alpha)) = 1$.

Definition 18 (*Logical consequence*) A formula β is called a *logical consequence* of a formula α iff for any formula γ such that α and β are subformulas of γ and for any model $\mathtt{Hol}$:

$$\mathtt{p}(\mathtt{Hol}^{\gamma}(\alpha)) \;\leq\; \mathtt{p}(\mathtt{Hol}^{\gamma}(\beta)).$$

This notion of logical consequence semantically characterizes a special example of logic that has been termed *holistic quantum computational logic*.

The concept of *model* $\mathtt{Hol}$ of the standard (unitary) quantum computational semantics can be naturally transformed into the notion of *pseudo-unitary model* $\widehat{\mathtt{Hol}}$. While $\mathtt{Hol}$ interprets the logical connectives occurring in the formulas of the language $\mathcal{L}$ as the corresponding unitary gates, $\widehat{\mathtt{Hol}}$ interprets the same connectives as the corresponding pseudo-unitary gates. Consider a formula α and suppose that $\mathtt{Hol}$ and $\widehat{\mathtt{Hol}}$ assign to the top-level of $STree^{\alpha}$ the same global meaning, represented by a density operator ρ of $\mathcal{H}^{\alpha}$. As expected, we will generally have:

$$\widehat{\mathtt{Hol}}(\alpha) \neq \mathtt{Hol}(\alpha).$$

In spite of this, one can prove that:

$$\mathtt{p}(\widehat{\mathtt{Hol}}(\alpha)) = \mathtt{p}(\mathtt{Hol}(\alpha)).$$

This is an immediate consequence of Theorem 2 and of our definition of model. Hence, we can conclude that the unitary and the pseudo-unitary holistic semantics determine the same concept of *logical consequence*. Holistic quantum computational logic can be equivalently characterized either by the unitary or by the pseudo-unitary semantics.

5 A Logical Characterization of Entanglement

In order to investigate the possibility of a logical characterization of entanglement it is useful to extend the standard quantum computational language $\mathcal{L}$ to a richer "mixed" language $\mathcal{L}^M$ that contains at the same time unitary connectives ($\neg$, $\top$, $\uplus$, $\sqrt{\neg}$, $\sqrt{id}$), pseudo-unitary connectives ($\widehat{\neg}$, $\widehat{\sqrt{\neg}}$) and a *Fredkin-connective* $\mathcal{F}^{(m,n,p)}$ for any $m, n, p \geq 1$. Semantically, $\mathcal{F}^{(m,n,p)}$ is interpreted as the corresponding Fredkin-gate $\mathtt{F}^{(m,n,p)}$).

The use of the Fredkin-connective $\mathcal{F}^{(m,n,p)}$ is governed by the following syntactical rule. For any $m, n, p \geq 1$ and for any sequence

$$(\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s, \gamma_1, \ldots, \gamma_t)$$

of formulas such that $At(\alpha_1) + \ldots + At(\alpha_r) = m$, $At(\beta_1) + \ldots + At(\beta_s) = n$, $At(\gamma_1) + \ldots + At(\gamma_t) = p$, the expression

$$\mathcal{F}^{(m,n,p)}(\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s, \gamma_1, \ldots, \gamma_t)$$

is a formula of $\mathcal{L}^M$. For instance, $\mathcal{F}^{(1,1,1)}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{f})$ is an example of a Fredkin-formula.

The concept of *model* of the language $\mathcal{L}^M$ is defined like in the case of $\mathcal{L}$, *mutatis mutandis*.

Consider now a bipartite state ρ (which describes a composite quantum system consisting of two subsystems) and let $\mathcal{C}(\rho)$ be the concurrence of ρ.[7] Is it possible to determine $\mathcal{C}(\rho)$ in terms of some meanings of particular formulas of the language $\mathcal{L}^M$? Let us first refer to the simplest situation, where ρ lives in the space $\mathcal{H}^{(2)}$. In such a case, ρ may represent the contextual meaning (with respect to a given model $\mathtt{Hol}$) of a pair of atomic formulas δ and θ that are subformulas of other formulas.

[7] We recall that the concurrence of ρ is defined as follows: $\mathcal{C}(\rho) := \inf\left\{\sum_i w_i \mathcal{C}(P_{|\psi_i\rangle}) : \rho = \sum_i w_i P_{|\psi_i\rangle}\right\}$, where $\mathcal{C}(P_{|\psi\rangle}) = \sqrt{2\left(1 - \sum_i \lambda_i^2\right)}$ and the numbers λ_i are eigenvalues of $Red^{(1)}_{[m,n]}(P_{|\psi\rangle})$ (or, equivalently, of $Red^{(2)}_{[m,n]}(P_{|\psi\rangle})$) (see Section 3 of the first part of this article).

We recall that for any ρ of $\mathcal{H}^{(2)}$, the concurrence of ρ satisfies the following relation[12]:

$$\mathcal{C}(\rho) = \max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\},$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the eigenvalues (in decreasing order) of the operator $\rho\left[{}^{\mathfrak{D}}(\widehat{\mathtt{NOT}}^{(1)} \otimes \widehat{\mathtt{NOT}}^{(1)})\rho\right]$. Hence, in order to estimate $\mathcal{C}(\rho)$, it is sufficient to determine the four numbers $\lambda_1, \lambda_1, \lambda_3, \lambda_4$. To this aim, we define four formulas $\eta_1,\ \eta_2,\ \eta_3,\ \eta_4$ and a model $\mathtt{Hol}$ (of $\mathcal{L}^M$) such that the four meanings $\mathtt{Hol}(\eta_1), \mathtt{Hol}(\eta_2), \mathtt{Hol}(\eta_3)$, $\mathtt{Hol}(\eta_4)$ allow us to determine the numbers $\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4$. The formula η_1 is defined as follows:

$$\eta_1 = \sqrt{id}\,\mathcal{F}^{(1,2,2)}(\mathcal{F}^{(2,2,1)}(\delta, \theta, \widehat{\neg}\delta, \widehat{\neg}\theta, \sqrt{id}\mathbf{f}))$$

(where **f** is the false formula).

Since the atomic complexity of η_1 is 5, all possible meanings of η_1 shall live in the space $\mathcal{H}^{(5)}$. The gate-tree of η_1 is the following sequence of gates (of the space $\mathcal{H}^{(5)}$):

$$\begin{aligned}
{}^{\mathfrak{D}}\mathtt{G}_4^{\eta_1} &= {}^{\mathfrak{D}}\left(\mathtt{I}^{(1)} \otimes \mathtt{I}^{(1)} \otimes \widehat{\mathtt{NOT}}^{(1)} \otimes \widehat{\mathtt{NOT}}^{(1)} \otimes \sqrt{\mathtt{I}}^{(1)}\right)\\
{}^{\mathfrak{D}}\mathtt{G}_3^{\eta_1} &= {}^{\mathfrak{D}}\mathtt{F}^{(2,2,1)}\\
{}^{\mathfrak{D}}\mathtt{G}_2^{\eta_1} &= {}^{\mathfrak{D}}\mathtt{F}^{(1,2,2)}\\
{}^{\mathfrak{D}}\mathtt{G}_1^{\eta_1} &= {}^{\mathfrak{D}}\sqrt{\mathtt{I}}^{(5)}
\end{aligned}$$

Semantically, it turns out that the contextual meanings of δ and of $\widehat{\neg}\delta$, of θ and of $\widehat{\neg}\theta$ shall be swapped when the last formula corresponds to a false register.

Consider now the following input for the gate-tree of η_1:

$$\rho \otimes \rho \otimes P_0.$$

This input gives rise to the following output:

$$\rho_1 = {}^{\mathfrak{D}}\mathtt{G}_1^{\eta_1}\left({}^{\mathfrak{D}}\mathtt{G}_2^{\eta_1}\left({}^{\mathfrak{D}}\mathtt{G}_3^{\eta_1}\left({}^{\mathfrak{D}}\mathtt{G}_4^{\eta_1}(\rho \otimes \rho \otimes P_0)\right)\right)\right).$$

The formula η_2 is obtained, in a similar way, by repeating two times the formulas' sequence $(\delta, \theta, \widehat{\neg}\delta, \widehat{\neg}\theta)$. Accordingly, we have:

$$\eta_2 = \sqrt{id}\,\mathcal{F}^{(1,2,6)}(\mathcal{F}^{(2,2,5)}(\mathcal{F}^{(5,2,2)}(\mathcal{F}^{(6,2,1)}(\delta, \theta, \widehat{\neg}\delta, \widehat{\neg}\theta, \delta, \theta, \widehat{\neg}\delta, \widehat{\neg}\theta, \sqrt{id}\mathbf{f})))).$$

In general, for $k \in \{1, 2, 3, 4\}$, we define η_k as follows:

$$\eta_k = \sqrt{id}\,\mathcal{F}^{(1,2,4k-2)}(\mathcal{F}^{(2,2,4k-3)}(\ldots$$
$$\ldots\mathcal{F}^{(1+4(k-1),2,2)}\mathcal{F}^{(2+4(k-1),2,1)}(\underbrace{\delta, \theta, \widehat{\neg}\delta, \widehat{\neg}\theta, \ldots, \delta, \theta, \widehat{\neg}\delta, \widehat{\neg}\theta}_{k}, \sqrt{id}\mathbf{f}))\ldots)).$$

Thus, for each η_k, the corresponding gate-tree

$$\left({}^{\mathfrak{D}}\mathtt{G}_1^{\eta_k},\ {}^{\mathfrak{D}}\mathtt{G}_2^{\eta_k}, \ldots,\ {}^{\mathfrak{D}}\mathtt{G}_{2k+1}^{\eta_k},\ {}^{\mathfrak{D}}\mathtt{G}_{2k+2}^{\eta_k}\right)$$

represents a quantum logical circuit whose inputs and outputs live in the space $\mathcal{H}^{(4k+1)}$.

Consider now, for each η_k, the input $\underbrace{\rho \otimes \ldots \otimes \rho}_{2k} \otimes P_0$, and let ρ_k be the corresponding output. One can easily show that there is a model $\mathtt{Hol}$ that satisfies the following conditions:

1. $\mathtt{Hol}(Level_{Top}^{\eta_k}) = \underbrace{\rho \otimes \ldots \otimes \rho}_{2k} \otimes P_0$, for each η_k;
2. $\mathtt{Hol}(\eta_k) = \rho_k$, for each η_k;
3. $\mathtt{Hol}^{\eta_1}((\delta, \theta)) = \mathtt{Hol}^{\eta_2}((\delta, \theta)) = \mathtt{Hol}^{\eta_3}((\delta, \theta)) = \mathtt{Hol}^{\eta_4}((\delta, \theta)) = \rho.$

Lemma 1 *Let* $\underbrace{\rho \otimes \ldots \otimes \rho}_{2k}$ *be a density operator of* $\mathcal{H}^{(4k)}$ *(with* $\rho \in \mathfrak{D}(\mathcal{H}^{(2)})$*). Consider the following number (which corresponds to a visibility-parameter):*

$$v = tr(\mathtt{SH}^{(4k)} \underbrace{\rho \otimes \ldots \otimes \rho}_{2k}) = tr(\rho^{2k}),$$

(where $\mathtt{SH}^{(4k)}$ *is the shift-gate).*[8] *We have:*

$$tr\left(P_0^{(4k+1)}\,{}^{\mathfrak{D}}\mathtt{G}_1^{\eta_k}\left(\ldots\,{}^{\mathfrak{D}}\mathtt{G}_{2k+1}^{\eta_k}\left({}^{\mathfrak{D}}\mathtt{I}^{(4k)} \otimes \sqrt{\mathtt{I}}^{(1)}\left(\underbrace{\rho \otimes \ldots \otimes \rho}_{2k} \otimes P_0\right)\right)\ldots\right)\right) = \frac{1+v}{2}.$$

The proof of Lemma 1 is based on [6], which describes a quantum network for direct estimations of both linear and non-linear functionals of any state.

Theorem 3 *For each* η_k*, the probability-value* $\mathtt{p}(\rho_k) = \mathtt{p}(\mathtt{Hol}(\eta_k))$ *satisfies the following relation:*

$$\mathtt{p}(\rho_k) = \frac{1 - tr(\overbrace{\rho\,[\,{}^{\mathfrak{D}}(\widehat{\mathtt{NOT}}^{(1)} \otimes \widehat{\mathtt{NOT}}^{(1)})\rho] \ldots \rho\,[\,{}^{\mathfrak{D}}(\widehat{\mathtt{NOT}}^{(1)} \otimes \widehat{\mathtt{NOT}}^{(1)})\rho]}^{k})}{2}.$$

Proof (Sketch) By lemma 1, $\mathtt{p}(\rho_k) = \mathrm{tr}\left(P_1^{(4k+1)}\rho_k\right) = \frac{1-v}{2}$. One can easily show that the Fredkin-gates realize a shift-operation ${}^{\mathfrak{D}}\mathtt{SH}^{(4k)}$ acting on

$$\overbrace{\rho \otimes [\,{}^{\mathfrak{D}}(\widehat{\mathtt{NOT}}^{(1)} \otimes \widehat{\mathtt{NOT}}^{(1)})\rho] \otimes \ldots \otimes \rho \otimes [\,{}^{\mathfrak{D}}(\widehat{\mathtt{NOT}}^{(1)} \otimes \widehat{\mathtt{NOT}}^{(1)})\rho]}^{k}$$

when the control state is $P_0^{(1)}$. Hence, by lemma 1,

$$v = \mathrm{tr}(\overbrace{\rho\,[\,{}^{\mathfrak{D}}(\widehat{\mathtt{NOT}}^{(1)} \otimes \widehat{\mathtt{NOT}}^{(1)})\rho] \ldots \rho\,[\,{}^{\mathfrak{D}}(\widehat{\mathtt{NOT}}^{(1)} \otimes \widehat{\mathtt{NOT}}^{(1)})\rho]}^{k}).$$

□

As a consequence we obtain:

$$\mathtt{p}(\rho_k) = \frac{1 - \sum_{i=1}^{4} \lambda_i^k}{2},$$

where $\lambda_1, \ldots, \lambda_4$ are the eigenvalues of the operator $\rho\,[\,{}^{\mathfrak{D}}(\widehat{\mathtt{NOT}}^{(1)} \otimes \widehat{\mathtt{NOT}}^{(1)})\rho]$.

Finally, let us consider the following four equations:

$$\begin{cases} 1 - 2\mathtt{p}(\rho_1) = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \\ 1 - 2\mathtt{p}(\rho_2) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 \\ 1 - 2\mathtt{p}(\rho_3) = \lambda_1^3 + \lambda_2^3 + \lambda_3^3 + \lambda_4^3 \\ 1 - 2\mathtt{p}(\rho_4) = \lambda_1^4 + \lambda_2^4 + \lambda_3^4 + \lambda_4^4 \end{cases}$$

[8] Notice that $\mathrm{tr}(\mathtt{SH}^{(4k)} \underbrace{\rho \otimes \ldots \otimes \rho}_{2k}) \neq \mathrm{tr}({}^{\mathfrak{D}}\mathtt{SH}^{(4k)} \underbrace{\rho \otimes \ldots \otimes \rho}_{2k}) = 1.$

This equation-system allows us to determine the numbers $\lambda_1, \ldots, \lambda_4$ in terms of the probability-values $p(\rho_1), \ldots, p(\rho_4)$.

We have obtained in this way a *logical protocol* that allows us to determine the concurrence of a density operator ρ (living in the space $\mathcal{H}^{(2)}$) in terms of the probability-values of the meanings of four formulas (under a model Hol that depends on ρ). This procedure can be generalized to any bipartite ρ, living in a space $\mathcal{H}^{(m+n)}$, by a convenient choice of formulas of the language $\mathcal{L}^M$.

The quantum logical circuit (we have described) can be realized by an interferometer with a controlled shift-gate between two Hadamard-gates. In this situation, the *visibility* of the interference-patterns of the *target-ancilla* is modified by the controlled shift-operation. An experiment with post-selection has been described in [10].

Let us finally discuss the following interesting question: given a bipartite state ρ (living in the space $\mathcal{H}^{(2n)}$) are there any epistemic agents $\mathfrak{a}$ and $\mathfrak{a}'$ (with truth-perspectives $\mathfrak{T}_a$ and $\mathfrak{T}_{a'}$, respectively), who can estimate the concurrence of ρ by using a unitary gate instead of the anti-unitary negation? The following theorem gives a positive answer to this question for the space $\mathcal{H}^{(2)}$. Since both the unitary and the anti-unitary negations (defined on the space $\mathcal{H}^{(n)}$) only act on the last element of any register $|x_1, \ldots, x_n\rangle$, a generalization of the theorem to the space $\mathcal{H}^{(2n)}$ can be obtained in a natural way.

Theorem 4 *For any $\rho \in \mathfrak{D}(\mathcal{H}^{(2)})$, there exists an infinite set* $\mathbf{T}$ *of different truth-perspectives such that for any $\mathfrak{T}_{\mathfrak{a}}, \mathfrak{T}_{\mathfrak{a}'} \in \mathbf{T}$:*

$$^{\mathfrak{D}}\left(\mathtt{NOT}^{(1)}_{\mathfrak{T}_{\mathfrak{a}}} \otimes \mathtt{NOT}^{(1)}_{\mathfrak{T}_{\mathfrak{a}'}}\right)\, \rho = {}^{\mathfrak{D}}\left(\widehat{\mathtt{NOT}}^{(1)} \otimes \widehat{\mathtt{NOT}}^{(1)}\right)\, \rho.$$

Proof Consider a circle C determined by the intersection of the Poincaré-Bloch sphere with a plane (for instance, the dashed circle depicted in Fig. 1 of Section 3). The circle C can be represented as the set of all vectors (corresponding to points of the sphere) that are orthogonal to a given vector

$$\overrightarrow{v} = (-\sin\varphi, \cos\varphi, 0).$$

Accordingly, any point of C corresponds to a vector $\overrightarrow{u}$ having the following form:

$$\overrightarrow{u} = r(\sin\omega\cos\varphi, \sin\omega\sin\varphi, \cos\omega).$$

The density operator $\rho_{\overrightarrow{u}}$, corresponding to the vector $\overrightarrow{u}$, will have the form:

$$\rho_{\overrightarrow{u}} = \frac{1}{2}\begin{pmatrix} 1 + r\cos\omega & re^{-i\varphi}\sin\omega \\ re^{i\varphi}\sin\omega & 1 - r\cos\omega \end{pmatrix},$$

where $\omega \in [0, 2\pi)$ and $r \in [0, 1]$.

Now we want to flip any point of C into its antipodal point, by using a unitary quantum operation (which acts on the corresponding density operator). Let us first define the following unitary operator:

$$\mathtt{U}^{(1)}_{\varphi} = \begin{pmatrix} 0 & -e^{-i\varphi} \\ e^{i\varphi} & 0 \end{pmatrix}$$

One can show that for any $\rho_{\overrightarrow{u}}$ (such that $\overrightarrow{u}$ corresponds to a point of C):

$$^{\mathfrak{D}}\mathtt{U}^{(1)}_{\varphi}\rho_{\overrightarrow{u}} = {}^{\mathfrak{D}}\widehat{\mathtt{NOT}}^{(1)}\rho_{\overrightarrow{u}}.$$

As we have learnt in Section 3, given a circle C, there is a truth-perspective $\mathfrak{T}$ such that:

$$^{\mathfrak{D}}\widehat{\mathtt{NOT}}^{(1)}\rho_{\overrightarrow{u}} = {}^{\mathfrak{D}}\mathtt{NOT}^{(1)}_{\mathfrak{T}}\rho_{\overrightarrow{u}}, \quad \text{for any vector } \overrightarrow{u} \text{ belonging to } C.$$

Consider now the infinite set **T** of truth-perspectives that have the following form:

$$\mathfrak{T}(\zeta) = \left[\cos\left(\frac{\zeta}{2}\right)\mathtt{I} - i\sin\left(\frac{\zeta}{2}\right)(-\sin\varphi\,\sigma_x + \cos\varphi\,\sigma_y)\right]\mathfrak{T}$$

(where σ_x, σ_y are the first and the second Pauli matrices). There are infinitely many $\zeta \in [0, 2\pi)$ such that:

$$\mathtt{NOT}^{(1)}_{\mathfrak{T}(\zeta)} = \mathtt{NOT}^{(1)}_{\mathfrak{T}}.$$

Hence, ${}^{\mathfrak{D}}\mathtt{NOT}^{(1)}_{\mathfrak{T}(\zeta)}\,\rho_{\overrightarrow{u}} = {}^{\mathfrak{D}}\widehat{\mathtt{NOT}}^{(1)}\,\rho_{\overrightarrow{u}}$, for any vector $\overrightarrow{u}$ in C.

Finally, consider a density operator $\rho \in \mathfrak{D}(\mathcal{H}^{(2)})$, whose reduced states $Red^{(1)}_{[1,1]}(\rho)$ and $Red^{(2)}_{[1,1]}(\rho)$ correspond to two points of a circle C. We can conclude that there are infinitely many pairs of truth-perspectives $(\mathfrak{T}_{\mathfrak{a}}, \mathfrak{T}_{\mathfrak{a}'})$, such that

$$^{\mathfrak{D}}\left(\mathtt{NOT}^{(1)}_{\mathfrak{T}_{\mathfrak{a}}} \otimes \mathtt{NOT}^{(1)}_{\mathfrak{T}_{\mathfrak{a}'}}\right)\,\rho = {}^{\mathfrak{D}}\left(\widehat{\mathtt{NOT}}^{(1)} \otimes \widehat{\mathtt{NOT}}^{(1)}\right)\,\rho.$$

□

In conclusion, the results presented in the first and in the second part of this article show how entanglement cannot be generally described by means of "simple" quantum logical tools. At the same time, a logical characterization of entanglement can be obtained in the framework of a somewhat "sophisticated" quantum computational semantics, where the basic logical connectives correspond either to unitary or to pseudo-unitary gates.

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