

HOLISTIC LOGICAL ARGUMENTS IN QUANTUM COMPUTATION

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Dedicated to Anatolij Dvurečenskij

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ABSTRACT. Quantum computational logics represent a logical abstraction from the circuit-theory in quantum computation. In these logics formulas are supposed to denote pieces of quantum information (qubits, quregisters or mixtures of quregisters), while logical connectives correspond to (quantum logical) gates that transform quantum information in a reversible way. The characteristic holistic features of the quantum theoretic formalism (which play an essential role in entanglement-phenomena) can be used in order to develop a *holistic* version of the quantum computational semantics. In contrast with the *compositional* character of most standard semantic approaches, *meanings* of formulas are here dealt with as *global* abstract objects that determine the *contextual meanings* of the formulas' components (from the *whole* to the *parts*). We present a survey of the most significant logical arguments that are valid or that are possibly violated in the framework of this semantics. Some logical features that may appear *prima facie* strange seem to reflect pretty well informal arguments that are currently used in our rational activity.

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1. Introduction

According to a common belief a basic aim of our use of languages is communicating some information. There are however diverging theories about the general concept of *information*. What does exactly mean *understanding* or *interpreting* the information expressed by a sentence α of a language \mathcal{L} ?

As is well known, the classical approaches to logic and to information theory are based on a simple idea: the *informational meaning* of a sentence is represented by a *bit*, which corresponds to a classical *truth-value* (either 1 or 0). At the same time, sequences of n bits (*registers*) represent possible *informational meanings* of sequences consisting of n sentences. Such a sharp and dichotomic view of information has been put in question by a number of non-classical theories. For instance,

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in the framework of the so called “fuzzy thinking”, *uncertainty*, *ambiguity* and *vagueness* are currently investigated by referring to *truth-degrees*, which have replaced the classical truth-values *truth* and *falsity*.

Quantum computational logics are based on a different idea: the information expressed by a sentence may be ambiguous because it is stored by a quantum object, which is governed by the indeterministic laws of quantum theory.¹ Accordingly, in these logics sentences are supposed to denote pieces of quantum information (*qubits*, *quregisters* or, more generally, *mixtures of quregisters*), while the logical connectives are interpreted as *unitary quantum operations* that transform pieces of quantum information in a reversible way. One obtains, in this way, a great variety of *logical operators*: some of them represent the “quantum informational counterparts” of the standard connectives (like negation, conjunction, disjunction); some others correspond to *genuine* quantum operations that may transform classical inputs into quantum uncertainties. In this framework, some fundamental quantum theoretic concepts, like *superposition* and *entanglement* (which have often been described as mysterious and potentially paradoxical), can be used as a “semantic resource” for a formal analysis of theoretic situations (even far from microphysics) where *ambiguity*, *holism* and *contextuality* play a relevant role. In this paper we will present a survey of the most significant logical arguments that are valid or that are possibly violated according to a *holistic* version of quantum computational logic. We will see how some semantic properties of this logic, which may appear *prima facie* somewhat strange, seem to reflect pretty well both quantum theoretic situations and informal arguments that are currently used in our rational activity.

2. The mathematical environment

It is expedient to recall some basic concepts of quantum computation that play an important role in the quantum computational semantics.² The general mathematical environment is the n -fold tensor product of the Hilbert space \mathbb{C}^2 :

$$\mathcal{H}^{(n)} := \underbrace{\mathbb{C}^2 \otimes \dots \otimes \mathbb{C}^2}_{n\text{-times}},$$

where all pieces of quantum information live. The elements $|1\rangle = (0, 1)$ and $|0\rangle = (1, 0)$ of the canonical orthonormal basis $B^{(1)}$ of \mathbb{C}^2 represent, in this framework, the two classical bits, which can be also regarded as the canonical truth-values *Truth* and *Falsity*, respectively. The canonical basis of $\mathcal{H}^{(n)}$ is the set

$$B^{(n)} = \left\{ |x_1\rangle \otimes \dots \otimes |x_n\rangle : |x_1\rangle, \dots, |x_n\rangle \in B^{(1)} \right\}.$$

As usual, we will briefly write $|x_1, \dots, x_n\rangle$ instead of $|x_1\rangle \otimes \dots \otimes |x_n\rangle$. By definition, a *quregister* is a unit vector of $\mathcal{H}^{(n)}$; while a *qubit* is a quregister of $\mathcal{H}^{(1)}$. Quregisters thus correspond to pure states, namely to maximal pieces of information about the quantum systems that are supposed to store a given amount of quantum information. We shall also make reference to *mixtures* of quregisters, represented by density operators ρ of $\mathcal{H}^{(n)}$. Of course, any quregister $|\psi\rangle$ corresponds to a special example of density operator: the projection operator $P_{|\psi\rangle}$ that projects over the closed subspace determined by $|\psi\rangle$. We will denote by $\mathfrak{D}(\mathcal{H}^{(n)})$ the set of all density operators of $\mathcal{H}^{(n)}$,

¹See, for instance, [4], [9], [7].

²See, for instance, [4], [10], [1].

while $\mathfrak{D} = \bigcup_n \{\mathfrak{D}(\mathcal{H}^{(n)})\}$ will represent the set of all possible pieces of quantum information, briefly called *qumixes*.

The choice of an orthonormal basis for the space \mathbb{C}^2 is, obviously, a matter of convention. One can consider infinitely many bases that are determined by the application of a unitary operator \mathfrak{T} to the elements of the canonical basis. From an intuitive point of view, we can think that the operator \mathfrak{T} gives rise to a change of *truth-perspective*. While in the classical case, the truth-values *Truth* and *Falsity* are identified with the two classical bits $|1\rangle$ and $|0\rangle$, assuming a different basis corresponds to a different idea of *Truth* and *Falsity*.³ Since any basis-change in \mathbb{C}^2 is determined by a unitary operator, we can identify a *truth-perspective* with a unitary operator \mathfrak{T} of \mathbb{C}^2 . We will write:

$$|1_{\mathfrak{T}}\rangle = \mathfrak{T}|1\rangle; \quad |0_{\mathfrak{T}}\rangle = \mathfrak{T}|0\rangle,$$

and we will assume that $|1_{\mathfrak{T}}\rangle$ and $|0_{\mathfrak{T}}\rangle$ represent, respectively, the truth-values *Truth* and *Falsity* of the truth-perspective \mathfrak{T} . The *canonical truth-perspective* is, of course, determined by the identity operator \mathbb{I} of \mathbb{C}^2 . We will indicate by $B_{\mathfrak{T}}^{(1)}$ the orthonormal basis determined by \mathfrak{T} ; while $B_{\mathbb{I}}^{(1)}$ will represent the canonical basis. From a physical point of view, we can suppose that each truth-perspective is associated to an apparatus that allows one to measure a given observable.

Any unitary operator \mathfrak{T} of $\mathcal{H}^{(1)}$ can be naturally extended to a unitary operator $\mathfrak{T}^{(n)}$ of $\mathcal{H}^{(n)}$ (for any $n \geq 1$):

$$\mathfrak{T}^{(n)}|x_1, \dots, x_n\rangle = \mathfrak{T}|x_1\rangle \otimes \dots \otimes \mathfrak{T}|x_n\rangle.$$

Accordingly, any choice of a unitary operator \mathfrak{T} of $\mathcal{H}^{(1)}$ determines an orthonormal basis $B_{\mathfrak{T}}^{(n)}$ for $\mathcal{H}^{(n)}$ such that:

$$B_{\mathfrak{T}}^{(n)} = \left\{ \mathfrak{T}^{(n)}|x_1, \dots, x_n\rangle : |x_1, \dots, x_n\rangle \in B_{\mathbb{I}}^{(n)} \right\}.$$

Instead of $\mathfrak{T}^{(n)}|x_1, \dots, x_n\rangle$ we will also write $|x_{1_{\mathfrak{T}}}, \dots, x_{n_{\mathfrak{T}}}\rangle$.

The elements of $B_{\mathfrak{T}}^{(1)}$ will be called the \mathfrak{T} -bits of $\mathcal{H}^{(1)}$; while the elements of $B_{\mathfrak{T}}^{(n)}$ will represent the \mathfrak{T} -registers of $\mathcal{H}^{(n)}$. On this ground the notions of *truth*, *falsity* and *probability* with respect to any truth-perspective \mathfrak{T} can be defined in a natural way.

DEFINITION 2.1 (\mathfrak{T} -true and \mathfrak{T} -false registers).

- $|x_{1_{\mathfrak{T}}}, \dots, x_{n_{\mathfrak{T}}}\rangle$ is a \mathfrak{T} -true register iff $|x_{n_{\mathfrak{T}}}\rangle = |1_{\mathfrak{T}}\rangle$;
- $|x_{1_{\mathfrak{T}}}, \dots, x_{n_{\mathfrak{T}}}\rangle$ is a \mathfrak{T} -false register iff $|x_{n_{\mathfrak{T}}}\rangle = |0_{\mathfrak{T}}\rangle$.

In other words, the \mathfrak{T} -truth-value of a \mathfrak{T} -register (which corresponds to a sequence of \mathfrak{T} -bits) is determined by its last element.

DEFINITION 2.2 (\mathfrak{T} -truth and \mathfrak{T} -falsity).

- The \mathfrak{T} -truth of $\mathcal{H}^{(n)}$ is the projection operator ${}^{\mathfrak{T}}P_1^{(n)}$ that projects over the closed subspace spanned by the set of all \mathfrak{T} - true registers;
- the \mathfrak{T} -falsity of $\mathcal{H}^{(n)}$ is the projection operator ${}^{\mathfrak{T}}P_0^{(n)}$ that projects over the closed subspace spanned by the set of all \mathfrak{T} - false registers.

In this way, truth and falsity are dealt with as mathematical representatives of possible physical properties. Accordingly, by applying the Born-rule, one can naturally define the probability-value of any qumix with respect to the truth-perspective \mathfrak{T} .

³Truth-perspectives play an important role in the case of *epistemic quantum computational logics*. See, for instance, [2] and [3].

DEFINITION 2.3 (\mathfrak{T} -Probability). For any $\rho \in \mathfrak{D}(\mathcal{H}^{(n)})$,

$$p_{\mathfrak{T}}(\rho) := \text{tr}({}^{\mathfrak{T}}P_1^{(n)}\rho),$$

where tr is the trace-functional.

We interpret $p_{\mathfrak{T}}(\rho)$ as the probability that the information ρ satisfies the \mathfrak{T} -Truth. In the particular case of qubits, we will obviously obtain:

$$p_{\mathfrak{T}}(a_0|0_{\mathfrak{T}}) + a_1|1_{\mathfrak{T}}) = |a_1|^2.$$

For any choice of a truth-perspective \mathfrak{T} , the set \mathfrak{D} of all density operators can be pre-ordered by a relation that is defined in terms of the probability-function $p_{\mathfrak{T}}$. In Section 4 we will see how this relation will play an important semantic role.

DEFINITION 2.4 (Preorder).

$$\rho \preceq_{\mathfrak{T}} \sigma \iff p_{\mathfrak{T}}(\rho) \leq p_{\mathfrak{T}}(\sigma).$$

When \mathfrak{T} is the canonical truth-perspective \mathbf{I} , we will also write: $P_1^{(n)}$, $P_0^{(n)}$, p , \preceq (instead of ${}^{\mathbf{I}}P_1^{(n)}$, ${}^{\mathbf{I}}P_0^{(n)}$, $p_{\mathbf{I}}$, $\preceq_{\mathbf{I}}$).

As is well known, *entanglement* represents one of the most crucial (and to a certain extent “mysterious”) feature of quantum theory. Consider a composite system $S = S_1 + \dots + S_t$ and its Hilbert space $\mathcal{H}^{(n)} = \mathcal{H}^{(n_1)} \otimes \dots \otimes \mathcal{H}^{(n_t)}$. Let $\rho \in \mathfrak{D}(\mathcal{H}^{(n)})$ be a state of S and let $i_1, \dots, i_r \in \{1, \dots, t\}$. The quantum theoretic formalism determines the *reduced state* of ρ with respect to the subsystem $S_{i_1} + \dots + S_{i_r}$. We will indicate this state by $\text{Red}_{[n_1, \dots, n_t]}^{(i_1, \dots, i_r)}(\rho)$.

It is expedient to recall a characteristic property of reduced states (described by the following Lemma).

LEMMA 2.1. *Let $\rho \in \mathfrak{D}(\mathcal{H}^{(m)} \otimes \mathcal{H}^{(p)})$. The reduced state $\text{Red}_{[m,p]}^{(1)}$ is the unique operation of $\mathfrak{D}(\mathcal{H}^{(m)} \otimes \mathcal{H}^{(p)})$ into $\mathfrak{D}(\mathcal{H}^{(m)})$ such that for any self-adjoint operator $A^{(m)}$ of $\mathcal{H}^{(m)}$ and for any $\rho \in \mathfrak{D}(\mathcal{H}^{(m)} \otimes \mathcal{H}^{(p)})$:*

$$\text{tr}((A^{(m)} \otimes \mathbf{I}^{(p)})\rho) = \text{tr}(A^{(m)} \text{Red}_{[m,p]}^{(1)}(\rho)).$$

A similar relation holds for the reduced state $\text{Red}_{[m,p]}^{(2)}$.

A characteristic situation that arises in entanglement-phenomena is the following: while the state of the global system is pure (a maximal information), the reduced states of some subsystems are mixtures (non-maximal pieces of information). Hence our information about the *whole* cannot be reconstructed as a function of our pieces of information about the *parts*. Although entanglement can be defined both for pure and for mixed states, in this article we will be only concerned with entangled quregisters.

DEFINITION 2.5 (*t-partite entangled quregister*). A quregister $|\psi\rangle$ of $\mathcal{H}^{(n)} = \mathcal{H}^{(n_1)} \otimes \dots \otimes \mathcal{H}^{(n_t)}$ is called a *t-partite entangled state* iff all reduced states $\text{Red}_{[n_1, \dots, n_t]}^{(1)}(P_{|\psi\rangle}), \dots, \text{Red}_{[n_1, \dots, n_t]}^{(t)}(P_{|\psi\rangle})$ are proper mixtures.

As a consequence a *t-partite entangled quregister* cannot be represented as a tensor product of the reduced states of its parts. When all reduced states $\text{Red}_{[n_1, \dots, n_t]}^{(i)}(P_{|\psi\rangle})$ are the quimix $\frac{1}{2^{n_i}}\mathbf{I}^{(n_i)}$ (which represents a perfect ambiguous information) one says that $|\psi\rangle$ is a *t-partite maximally entangled state*.

DEFINITION 2.6 (Entangled quregister with respect to some parts). A quregister $|\psi\rangle$ of $\mathcal{H}^{(n)} = \mathcal{H}^{(n_1)} \otimes \dots \otimes \mathcal{H}^{(n_t)}$ is called *entangled* with respect to its parts labelled by the indices i_1, \dots, i_r (with $1 \leq i_1, \dots, i_r \leq t$) iff the reduced states $\text{Red}_{[n_1, \dots, n_t]}^{(i_1)}(P_{|\psi\rangle}), \dots, \text{Red}_{[n_1, \dots, n_t]}^{(i_r)}(P_{|\psi\rangle})$ are proper mixtures.

Since the notion of reduced state is independent of the choice of a particular basis, it turns out that the status of *t-partite entangled quregister*, *maximally entangled quregister* and *entangled quregister with respect to some parts* is invariant under changes of truth-perspective.

Example 1.

- The quregister

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0, 0, 0\rangle + |1, 1, 1\rangle)$$

is a 3-partite maximally entangled quregister of $\mathcal{H}^{(3)}$;

- the quregister

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0, 0, 0\rangle + |1, 1, 0\rangle)$$

is an entangled quregister of $\mathcal{H}^{(3)}$ with respect to its first and second part.

3. Quantum logical gates and the holistic conjunction

As is well known, quantum information is processed by *quantum logical gates* (briefly, *gates*): unitary operators that transform quregisters into quregisters in a reversible way. Let us recall the definition of some gates that play a special role both from the computational and from the logical point of view.

DEFINITION 3.1 (The negation). For any $n \geq 1$, the *negation* on $\mathcal{H}^{(n)}$ is the linear operator $\text{NOT}^{(n)}$ such that, for every element $|x_1, \dots, x_n\rangle$ of the canonical basis,

$$\text{NOT}^{(n)}|x_1, \dots, x_n\rangle = |x_1, \dots, x_{n-1}\rangle \otimes |1 - x_n\rangle.$$

In particular, we obtain:

$$\text{NOT}^{(1)}|0\rangle = |1\rangle; \quad \text{NOT}^{(1)}|1\rangle = |0\rangle,$$

according to the classical truth-table of negation.

DEFINITION 3.2 (The Toffoli-gate). For any $m, n, p \geq 1$, the *Toffoli-gate* is the linear operator $\mathbf{T}^{(m, n, p)}$ defined on $\mathcal{H}^{(m+n+p)}$ such that, for every element $|x_1, \dots, x_m\rangle \otimes |y_1, \dots, y_n\rangle \otimes |z_1, \dots, z_p\rangle$ of the canonical basis,

$$\mathbf{T}^{(m, n, p)}|x_1, \dots, x_m, y_1, \dots, y_n, z_1, \dots, z_p\rangle = |x_1, \dots, x_m, y_1, \dots, y_n, z_1, \dots, z_{p-1}\rangle \otimes |x_m y_n \hat{+} z_p\rangle,$$

where $\hat{+}$ represents the addition modulo 2.

The following Lemma asserts a characteristic property of the Toffoli-gate (which turns out to be useful from the computational point of view).

LEMMA 3.1. ([5])

$$\mathbf{T}^{(m, n, 1)} = [(\mathbf{I}^{(m+n)} - P_1^{(m)} \otimes P_1^{(n)}) \otimes \mathbf{I}^{(1)}] + [P_1^{(m)} \otimes P_1^{(n)} \otimes \text{NOT}^{(1)}].$$

DEFINITION 3.3 (The XOR-gate). For any $m, n \geq 1$, the *XOR-gate* is the linear operator $\text{XOR}^{(m,n)}$ defined on $\mathcal{H}^{(m+n)}$ such that, for every element $|x_1, \dots, x_m\rangle \otimes |y_1, \dots, y_n\rangle$ of the canonical basis,

$$\text{XOR}^{(m,n)}|x_1, \dots, x_m, y_1, \dots, y_n\rangle = |x_1, \dots, x_m, y_1, \dots, y_{n-1}\rangle \otimes |x_m \hat{+} y_n\rangle.$$

DEFINITION 3.4 (The Hadamard-gate). For any $n \geq 1$, the *Hadamard-gate* on $\mathcal{H}^{(n)}$ is the linear operator $\sqrt{\mathbb{I}}^{(n)}$ such that for every element $|x_1, \dots, x_n\rangle$ of the canonical basis:

$$\sqrt{\mathbb{I}}^{(n)}|x_1, \dots, x_n\rangle = |x_1, \dots, x_{n-1}\rangle \otimes \frac{1}{\sqrt{2}}((-1)^{x_n}|x_n\rangle + |1 - x_n\rangle).$$

In particular we obtain:

$$\sqrt{\mathbb{I}}^{(1)}|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle); \quad \sqrt{\mathbb{I}}^{(1)}|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

Hence, $\sqrt{\mathbb{I}}^{(1)}$ transforms bits into genuine qubits.

DEFINITION 3.5 (The square root of NOT). For any $n \geq 1$, the *square root of NOT* on $\mathcal{H}^{(n)}$ is the linear operator $\sqrt{\text{NOT}}^{(n)}$ such that for every element $|x_1, \dots, x_n\rangle$ of the canonical basis:

$$\sqrt{\text{NOT}}^{(n)}|x_1, \dots, x_n\rangle = |x_1, \dots, x_{n-1}\rangle \otimes \left(\frac{1-i}{2}|x_n\rangle + \frac{1+i}{2}|1-x_n\rangle \right),$$

where $i = \sqrt{-1}$.

All gates can be naturally transposed from the canonical truth-perspective to any truth-perspective \mathfrak{T} . Let $G^{(n)}$ be any gate defined with respect to the canonical truth-perspective. The *twin-gate* $G_{\mathfrak{T}}^{(n)}$, defined with respect to the truth-perspective \mathfrak{T} , is determined as follows:

$$G_{\mathfrak{T}}^{(n)} := \mathfrak{T}^{(n)} G^{(n)} \mathfrak{T}^{(n)\dagger},$$

where $\mathfrak{T}^{(n)\dagger}$ is the adjoint of \mathfrak{T} .

All \mathfrak{T} -gates can be canonically extended to the set \mathfrak{D} of all qumixes. Let $G_{\mathfrak{T}}$ be any gate defined on $\mathcal{H}^{(n)}$. The corresponding *qumix gate* (also called *unitary quantum operation*) $\mathfrak{D}G_{\mathfrak{T}}$ is defined as follows for any $\rho \in \mathfrak{D}(\mathcal{H}^{(n)})$:

$$\mathfrak{D}G_{\mathfrak{T}}\rho = G_{\mathfrak{T}}\rho G_{\mathfrak{T}}^\dagger.$$

For the sake simplicity, also the qumix gates $\mathfrak{D}G_{\mathfrak{T}}$ will be briefly called *gates*.

The Toffoli-gate $\mathfrak{D}T_{\mathfrak{T}}^{(m,n,p)}$ allows us to define a reversible operation $\text{AND}_{\mathfrak{T}}^{(m,n)}$ that represents a *holistic conjunction*.

DEFINITION 3.6 (The holistic conjunction). For any $m, n \geq 1$ the *holistic conjunction* $\text{AND}_{\mathfrak{T}}^{(m,n)}$ with respect to the truth-perspective \mathfrak{T} is defined as follows for any qumix $\rho \in \mathfrak{D}(\mathcal{H}^{(m+n)})$:

$$\text{AND}_{\mathfrak{T}}^{(m,n)}(\rho) := \mathfrak{D}T_{\mathfrak{T}}^{(m,n,1)}(\rho \otimes \mathfrak{T}P_0^{(1)}),$$

where the \mathfrak{T} -falsity $\mathfrak{T}P_0^{(1)}$ plays the role of an *ancilla*.

When $\mathfrak{T} = \mathbb{I}$, we will write $\text{AND}^{(m,n)}$ (instead of $\text{AND}_{\mathbb{I}}^{(m,n)}$).

It is worth-while noticing that generally

$$\text{AND}_{\mathfrak{T}}^{(m,n)}(\rho) \neq \text{AND}_{\mathfrak{T}}^{(m,n)}(\text{Red}_{[m,n]}^{(1)}(\rho) \otimes \text{Red}_{[m,n]}^{(2)}(\rho)).$$

Roughly, we might say that the holistic conjunction defined on a global information consisting of two parts does not generally coincide with the conjunction of the two separate parts. As an

example, we can consider the following qumix (which corresponds to a maximally entangled pure state):

$$\rho = P_{\frac{1}{\sqrt{2}}(|0,0\rangle+|1,1\rangle)}.$$

We have:

$$\text{AND}^{(1,1)}(\rho) = \mathfrak{D}_{\mathbb{T}^{(1,1,1)}}\left(P_{\frac{1}{\sqrt{2}}(|0,0\rangle+|1,1\rangle)} \otimes P_0^{(1)}\right) = P_{\frac{1}{\sqrt{2}}(|0,0,0\rangle+|1,1,1\rangle)},$$

which also represents a maximally entangled quregister.

At the same time we have:

$$\text{AND}^{(1,1)}\left(\text{Red}_{[1,1]}^{(1)}(\rho) \otimes \text{Red}_{[1,1]}^{(2)}(\rho)\right) = \text{AND}^{(1,1)}\left(\frac{1}{2}\mathbb{I}^{(1)} \otimes \frac{1}{2}\mathbb{I}^{(1)}\right),$$

which is a proper mixture.

Furthermore, we have:

$$\mathfrak{p}(\text{AND}^{(1,1)}(\rho)) = \frac{1}{2}; \quad \mathfrak{p}\left(\text{AND}^{(1,1)}\left(\text{Red}_{[1,1]}^{(1)}(\rho) \otimes \text{Red}_{[1,1]}^{(2)}(\rho)\right)\right) = \frac{1}{4}.$$

We will now investigate some interesting probabilistic properties of the holistic conjunction (illustrated by the following Theor. 3.1 and Theor. 3.2).

Let us first recall that the set of all projection operators of a Hilbert space $\mathcal{H}^{(n)}$ is partially ordered by the following relation:

$$P \leq Q \iff PQ = P.$$

We have: $P \leq Q$ iff $\text{tr}(P\rho) \leq \text{tr}(Q\rho)$ for any $\rho \in \mathfrak{D}(\mathcal{H}^{(n)})$.

LEMMA 3.2.

- (1) $P_1^{(m)} \otimes P_1^{(n)} \leq P_1^{(m)} \otimes \mathbb{I}^{(n)}$.
- (2) $P_1^{(m)} \otimes P_1^{(n)} \leq \mathbb{I}^{(m)} \otimes P_1^{(n)}$.
- (3) For any $\rho \in \mathfrak{D}(\mathcal{H}^{(m)} \otimes \mathcal{H}^{(n)})$:
 - (a) $\mathfrak{p}(\rho) = \mathfrak{p}(\text{Red}_{[m+n-1,1]}^{(2)}(\rho))$;
 - (b) $\text{tr}((P_1^{(m)} \otimes \mathbb{I}^{(n)})\rho) = \text{tr}(P_1^{(1)} \text{Red}_{[m-1,1,n]}^{(2)}(\rho))$.

Proof.

$$(1) (P_1^{(m)} \otimes P_1^{(n)})(P_1^{(m)} \otimes \mathbb{I}^{(n)}) = (P_1^{(m)} P_1^{(m)}) \otimes (P_1^{(n)} \mathbb{I}^{(n)}) = P_1^{(m)} \otimes P_1^{(n)}.$$

(2) Similar to (1).

$$(3)(a) \mathfrak{p}(\rho) = \text{tr}(P_1^{(m+n)}\rho) = \text{tr}((\mathbb{I}^{(m+n-1)} \otimes P_1^{(1)})\rho) = \text{tr}(P_1^{(1)} \text{Red}_{[m+n-1,1]}^{(2)}(\rho)).$$

$$= \mathfrak{p}(\text{Red}_{[m+n-1,1]}^{(2)}(\rho))$$

$$(3)(b) \text{tr}((P_1^{(m)} \otimes \mathbb{I}^{(n)})\rho) = \text{tr}((\mathbb{I}^{(m-1)} \otimes P_1^{(1)} \otimes \mathbb{I}^{(n)})\rho) = \text{tr}((P_1^{(1)} \otimes \mathbb{I}^{(n)}) \text{Red}_{[m-1,1,n]}^{(2,3)}(\rho)).$$

$$= \text{tr}(P_1^{(1)} \text{Red}_{[m-1,1,n]}^{(2)}(\rho))$$

□

THEOREM 3.1. For any $\rho \in \mathfrak{D}(\mathcal{H}^{(m+n)})$, $\mathfrak{p}(\text{AND}^{(m,n)}(\rho)) = \text{tr}((P_1^{(m)} \otimes P_1^{(n)})\rho)$.

Proof. By definition of $\text{AND}^{(m,n)}$ and by Lemma 3.1:

$$\begin{aligned}
\text{AND}^{(m,n)}(\rho) &= \mathbf{T}^{(m,n,1)}(\rho \otimes P_0^{(1)})\mathbf{T}^{(m,n,1)} \\
&= [(\mathbf{I}^{(m+n)} - P_1^{(m)} \otimes P_1^{(n)}) \otimes \mathbf{I}^{(1)}](\rho \otimes P_0^{(1)})[(\mathbf{I}^{(m+n)} - P_1^{(m)} \otimes P_1^{(n)}) \otimes \mathbf{I}^{(1)}] \\
&\quad + [P_1^{(m)} \otimes P_1^{(n)} \otimes \text{NOT}^{(1)}](\rho \otimes P_0^{(1)})[P_1^{(m)} \otimes P_1^{(n)} \otimes \text{NOT}^{(1)}].
\end{aligned}$$

One can easily see that

$$P_1^{(m+n+1)}(\mathbf{I}^{(m+n)} - P_1^{(m)} \otimes P_1^{(n)}) \otimes \mathbf{I}^{(1)}(\rho \otimes P_0^{(1)})(\mathbf{I}^{(m+n)} - P_1^{(m)} \otimes P_1^{(n)}) \otimes \mathbf{I}^{(1)}$$

is the null projection operator. Consequently:

$$\begin{aligned}
\mathbf{p}(\text{AND}^{(m,n)}(\rho)) &= \text{tr}(P_1^{(m+n+1)}(P_1^{(m)} \otimes P_1^{(n)} \otimes \text{NOT}^{(1)})(\rho \otimes P_0^{(1)})(P_1^{(m)} \otimes P_1^{(n)} \otimes \text{NOT}^{(1)})) \\
&= \text{tr}(P_1^{(m+n+1)}((P_1^{(m)} \otimes P_1^{(n)})\rho(P_1^{(m)} \otimes P_1^{(n)})) \otimes \text{NOT}^{(1)}P_0^{(1)}\text{NOT}^{(1)}) \\
&= \text{tr}((P_1^{(m)} \otimes P_1^{(n)})\rho(P_1^{(m)} \otimes P_1^{(n)})) \text{tr}(P_1^{(1)}P_1^{(1)}) \\
&= \text{tr}((P_1^{(m)} \otimes P_1^{(n)})\rho).
\end{aligned}$$

□

THEOREM 3.2. For any $\rho \in \mathfrak{D}(\mathcal{H}^{(m)} \otimes \mathcal{H}^{(n)})$:

$$\mathbf{p}(\text{AND}^{(m,n)}(\rho)) \leq \mathbf{p}(\text{Red}_{[m,n]}^{(1)}(\rho)) \quad \text{and} \quad \mathbf{p}(\text{AND}^{(m,n)}(\rho)) \leq \mathbf{p}(\text{Red}_{[m,n]}^{(2)}(\rho)).$$

Proof. Let $\rho \in \mathfrak{D}(\mathcal{H}^{(m+n)})$. By Lemma 3.2 (1) we have:

$$P_1^{(m)} \otimes P_1^{(n)} \leq P_1^{(m)} \otimes \mathbf{I}^{(n)}.$$

Hence, by Lemma 3.2 (3):

$$\text{tr}((P_1^{(m)} \otimes P_1^{(n)})\rho) \leq \text{tr}((P_1^{(m)} \otimes \mathbf{I}^{(n)})\rho) = \text{tr}(P_1^{(1)} \text{Red}_{[m-1,1,n]}^{(2)}(\rho)) = \mathbf{p}(\text{Red}_{[m,n]}^{(1)}(\rho)).$$

Since $\mathbf{p}(\text{AND}^{(m,n)}(\rho)) = \text{tr}((P_1^{(m)} \otimes P_1^{(n)})\rho)$ (by Theorem 3.1), we obtain:

$$\mathbf{p}(\text{AND}^{(m,n)}(\rho)) \leq \mathbf{p}(\text{Red}_{[m,n]}^{(1)}(\rho)).$$

In a similar way one can prove that:

$$\mathbf{p}(\text{AND}^{(m,n)}(\rho)) \leq \mathbf{p}(\text{Red}_{[m,n]}^{(2)}(\rho)).$$

□

Theorems 3.1 and 3.2 (which have been proved for the canonical holistic conjunctions $\text{AND}^{(m,n)}$) can be easily generalized to any $\text{AND}_{\mathfrak{T}}^{(m,n)}$ (where \mathfrak{T} is any truth-perspective).

4. A Holistic quantum computational semantics

Let us first present the syntactical basis for our semantics. The linguistic framework is a *quantum computational language* \mathcal{L} , whose alphabet contains atomic formulas (say, “the spin-value in the x -direction is up”), including two privileged formulas \mathbf{t} and \mathbf{f} that represent the truth-values *Truth* and *Falsity*, respectively. The connectives of \mathcal{L} correspond to some gates that have a special logical and computational interest: the negation \neg (corresponding to the gate *negation*), a ternary connective \mathbf{T} (corresponding to the *Toffoli-gate*), the exclusive disjunction \oplus (corresponding to *XOR*), the square root of the identity $\sqrt{\text{id}}$ (corresponding to the *Hadamard-gate*), the square root of

negation $\sqrt{\neg}$ (corresponding to the gate *square root of NOT*). The notion of *formula* (or *sentence*) of \mathcal{L} is inductively defined (in the expected way). Accordingly, if α, β, γ are formulas, then the expressions $\neg\alpha, \sqrt{\text{id}}\alpha, \sqrt{\neg}\alpha, \top(\alpha, \beta, \gamma), \alpha \uplus \beta$ are formulas.

Recalling the definition of the holistic conjunction $\text{AND}^{(m,n)}$, it is useful to introduce a binary logical conjunction \wedge by means of the following metalinguistic definition:

$$\alpha \wedge \beta := \top(\alpha, \beta, \mathbf{f})$$

(where the false formula \mathbf{f} plays the role of a *syntactical ancilla*).

On this basis, a (binary) inclusive disjunction is (metalinguistically) defined via de Morgan-law:

$$\alpha \vee \beta := \neg(\neg\alpha \wedge \neg\beta).$$

The connectives \neg, \wedge, \vee and \uplus will be also termed *quantum computational Boolean connectives*; while $\sqrt{\text{id}}$ and $\sqrt{\neg}$ represent *genuine quantum computational connectives*. A formula that contains at most Boolean connectives is called a *Boolean formula* of \mathcal{L} .

In the following we will use $\mathbf{q}, \mathbf{q}_1, \dots$ as metavariables for atomic formulas, while $\alpha, \beta, \gamma, \dots$ will represent generic formulas.

DEFINITION 4.1 (The atomic complexity of a formula). The atomic complexity $\text{At}(\alpha)$ of a formula α is the number of occurrences of atomic formulas in α .

For instance, $\text{At}(\top(\mathbf{q}, \mathbf{q}, \mathbf{f})) = 3$. The notion of atomic complexity plays an important semantic role. As we will see, the meaning of any formula whose atomic complexity is n shall live in the domain $\mathfrak{D}(\mathcal{H}^n)$. For this reason, $\mathcal{H}^{\text{At}(\alpha)}$ (briefly indicated by \mathcal{H}^α) will be also called the *semantic space* of α .

Any formula α can be naturally decomposed into its parts, giving rise to a special configuration called the *syntactical tree* of α (indicated by STree^α). Roughly, STree^α can be represented as a finite sequence of *levels*:

$$\begin{array}{c} \text{Level}_h^\alpha \\ \dots\dots \\ \text{Level}_1^\alpha \end{array}$$

where:

- each Level_i^α (with $1 \leq i \leq h$) is a sequence $(\beta_1, \dots, \beta_m)$ of subformulas of α ;
- the *bottom level* Level_1^α is (α) ;
- the *top level* Level_h^α is the sequence $(\mathbf{q}_1, \dots, \mathbf{q}_r)$, where $\mathbf{q}_1, \dots, \mathbf{q}_r$ are the atomic occurrences in α ;
- for any i (with $1 \leq i < h$), $\text{Level}_{i+1}^\alpha$ is the sequence obtained by dropping the *principal connective* in all molecular formulas occurring at Level_i^α , and by repeating all the atomic sentences that occur at Level_i^α .

By *Height* of α (indicated by $\text{Height}(\alpha)$) we mean the number h of levels of the syntactical tree of α .

As an example, consider the following formula:

$$\alpha = \neg \top(\mathbf{q}, \neg\mathbf{q}, \mathbf{f}) = \neg(\mathbf{q} \wedge \neg\mathbf{q}),$$

which represents an instance of the non-contradiction principle.

The syntactical tree of α is the following sequence of levels:

$$\begin{aligned} \text{Level}_4^\alpha &= (\mathbf{q}, \mathbf{q}, \mathbf{f}) \\ \text{Level}_3^\alpha &= (\mathbf{q}, \neg\mathbf{q}; \mathbf{f}) \\ \text{Level}_2^\alpha &= (\top(\mathbf{q}, \neg\mathbf{q}, \mathbf{f})) \\ \text{Level}_1^\alpha &= (\neg \top(\mathbf{q}, \neg\mathbf{q}, \mathbf{f})) \end{aligned}$$

Clearly, $\text{Height}(\alpha) = 4$.

For any choice of a truth-perspective \mathfrak{T} , the syntactical tree of any formula α uniquely determines a sequence of gates, all defined on the semantic space of α . As an example, consider again the formula $\alpha = \neg \top(\mathbf{q}, \neg\mathbf{q}, \mathbf{f})$. In the syntactical tree of α the third level has been obtained from the fourth level by repeating the first occurrence of \mathbf{q} , by negating the second occurrence of \mathbf{q} and by repeating \mathbf{f} , while the second and the first level have been obtained by applying, respectively, the connectives \top and \neg to formulas occurring at the levels immediately above.

Accordingly, one can say that, for any choice of a truth-perspective \mathfrak{T} , the syntactical tree of α uniquely determines the following sequence consisting of three gates, all defined on the semantic space of α :

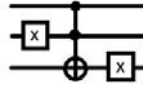
$$\left(\mathfrak{D}I_{\mathfrak{T}}^{(1)} \otimes \mathfrak{D}NOT_{\mathfrak{T}}^{(1)} \otimes \mathfrak{D}I_{\mathfrak{T}}^{(1)}, \quad \mathfrak{D}T_{\mathfrak{T}}^{(1,1,1)}, \quad \mathfrak{D}NOT_{\mathfrak{T}}^{(3)} \right).$$

Such a sequence is called the \mathfrak{T} -gate tree of α . This procedure can be naturally generalized to any formula α . The general form of the \mathfrak{T} -gate tree of α will be:

$$\left(\mathfrak{D}G_{\mathfrak{T}(h-1)}^\alpha, \dots, \mathfrak{D}G_{\mathfrak{T}(1)}^\alpha \right),$$

where h is the Height of α .

From an intuitive point of view, any formula α of \mathcal{L} can be regarded as a synthetic logical description of a quantum circuit that may assume as inputs qumixes living in the semantic space of α . For instance, the circuit described by $\alpha = \neg \top(\mathbf{q}, \neg\mathbf{q}, \mathbf{f})$ can be represented as follows:



Thus, \mathcal{L} -formulas turn out to have a characteristic *dynamic* character, representing systems of *computational actions*.

Now the holistic semantics comes into play.⁴ The intuitive idea can be sketched as follows. For any choice of a truth-perspective, a *holistic model* of the language \mathcal{L} assigns to any formula α a *global informational meaning* that lives in \mathcal{H}^α (the semantic space of α). This meaning determines the *contextual meanings* of all subformulas of α (from the whole to the parts!). It may happen that one and the same model assigns to a given formula α different contextual meanings in different contexts.

Before defining the concept of model, it is expedient to introduce the weaker notion of *holistic map* for the language \mathcal{L} .

DEFINITION 4.2 (Holistic map). A *holistic map* for \mathcal{L} (associated to a truth-perspective \mathfrak{T}) is a map $\text{Hol}_{\mathfrak{T}}$ that assigns a meaning $\text{Hol}_{\mathfrak{T}}(\text{Level}_i^\alpha)$ to each level of the syntactical tree of α , for any formula α . This meaning is a qumix living in the semantic space of α .

⁴See [6] and [2].

Given a formula γ , any holistic map $\text{Hol}_{\mathfrak{T}}$ determines the *contextual meaning*, with respect to the context $\text{Hol}_{\mathfrak{T}}(\gamma)$, of any occurrence of a subformula β in γ . This contextual meaning can be defined, in a natural way, by using the notion of *reduced state*.

DEFINITION 4.3 (Contextual meaning). Consider a formula γ such that $Level_i^\gamma = (\beta_{i_1}, \dots, \beta_{i_r})$. We have: $\mathcal{H}^\gamma = \mathcal{H}^{\beta_{i_1}} \otimes \dots \otimes \mathcal{H}^{\beta_{i_r}}$. Let $\text{Hol}_{\mathfrak{T}}$ be a holistic map. The *contextual meaning* of the occurrence β_{i_j} with respect to the context $\text{Hol}_{\mathfrak{T}}(\gamma)$ is defined as follows:

$$\text{Hol}_{\mathfrak{T}}^\gamma(\beta_{i_j}) := \text{Red}_{[\text{At}(\beta_{i_1}), \dots, \text{At}(\beta_{i_r})]}^j(\text{Hol}_{\mathfrak{T}}(Level_i(\gamma))).$$

Of course, we obtain:

$$\text{Hol}_{\mathfrak{T}}^\gamma(\gamma) = \text{Hol}_{\mathfrak{T}}(\gamma).$$

A holistic map $\text{Hol}_{\mathfrak{T}}$ is called *normal for a formula* γ iff for any subformula β of γ , $\text{Hol}_{\mathfrak{T}}$ assigns the same contextual meaning to all occurrences of β in the syntactical tree of γ . In other words:

$$\text{Hol}_{\mathfrak{T}}^\gamma(\beta_{i_j}) = \text{Hol}_{\mathfrak{T}}^\gamma(\beta_{u_v}),$$

where β_{i_j} and β_{u_v} are two occurrences of β in $STree^\gamma$.

A *normal holistic map* is a holistic map $\text{Hol}_{\mathfrak{T}}$ that is normal for all formulas γ .

DEFINITION 4.4 (Compositional holistic map). Consider a formula α such that $Level_h^\alpha = (\mathbf{q}_1, \dots, \mathbf{q}_r)$, while the \mathfrak{T} -gate tree of α is $({}^{\mathfrak{D}}G_{\mathfrak{T}(h-1)}^\alpha, \dots, {}^{\mathfrak{D}}G_{\mathfrak{T}(1)}^\alpha)$. A holistic map $\text{Hol}_{\mathfrak{T}}$ is called *compositional with respect to* α iff the following conditions are satisfied:

- (1) $\text{Hol}_{\mathfrak{T}}(Level_h^\alpha) = \text{Hol}_{\mathfrak{T}}^\alpha(\mathbf{q}_1) \otimes \dots \otimes \text{Hol}_{\mathfrak{T}}^\alpha(\mathbf{q}_r)$.
- (2) $\text{Hol}_{\mathfrak{T}}(Level_i^\alpha) = {}^{\mathfrak{D}}G_{\mathfrak{T}(i)}^\alpha(\text{Hol}_{\mathfrak{T}}(Level_{i+1}^\alpha))$, for any i (with $1 \leq i < h$).

LEMMA 4.1. *Any holistic map $\text{Hol}_{\mathfrak{T}}$ that is compositional with respect to the formula α satisfies the following conditions:*

- (1) *If $Level_i^\alpha = (\beta_{i_1}, \dots, \beta_{i_r})$, then*

$$\text{Hol}_{\mathfrak{T}}(Level_i^\alpha) = \text{Hol}_{\mathfrak{T}}^\alpha(\beta_{i_1}) \otimes \dots \otimes \text{Hol}_{\mathfrak{T}}^\alpha(\beta_{i_r}),$$

for any i such that $1 \leq i \leq \text{Height}(\alpha)$.

- (2) *$\text{Hol}_{\mathfrak{T}}$ is a normal holistic map for α .*

Proof.

- (1) By definition of compositional holistic map and by induction on i .
- (2) By definition of compositional holistic map and by (1). □

We can now define the concept of *holistic model* of the language \mathcal{L} .

DEFINITION 4.5 (Holistic model). A *holistic model* of \mathcal{L} is a normal holistic map $\text{Hol}_{\mathfrak{T}}$ that satisfies the following conditions for any formula α .

- (1) Let $({}^{\mathfrak{D}}G_{\mathfrak{T}(h-1)}^\alpha, \dots, {}^{\mathfrak{D}}G_{\mathfrak{T}(1)}^\alpha)$ be the \mathfrak{T} -gate tree of α and let $1 \leq i < h$. Then,

$$\text{Hol}_{\mathfrak{T}}(Level_i^\alpha) = {}^{\mathfrak{D}}G_{\mathfrak{T}(i)}^\alpha(\text{Hol}_{\mathfrak{T}}(Level_{i+1}^\alpha)).$$

In other words the meaning of each level (different from the top level) is obtained by applying the corresponding gate to the meaning of the level that occurs immediately above.

(2) Let $Level_i^\alpha = (\beta_{i_1}, \dots, \beta_{i_r})$. Then,

$$\beta_{i_j} = \mathbf{f} \implies \mathbf{Hol}_{\mathfrak{T}}^\alpha(\mathbf{f}) = \text{Red}_{[\text{At}(\beta_{i_1}), \dots, \text{At}(\beta_{i_r})]}^j(\mathbf{Hol}_{\mathfrak{T}}(Level_i^\alpha)) = {}_{\mathfrak{T}}P_0^{(1)};$$

$$\beta_{i_j} = \mathbf{t} \implies \mathbf{Hol}_{\mathfrak{T}}^\alpha(\mathbf{t}) = \text{Red}_{[\text{At}(\beta_{i_1}), \dots, \text{At}(\beta_{i_r})]}^j(\mathbf{Hol}_{\mathfrak{T}}(Level_i^\alpha)) = {}_{\mathfrak{T}}P_1^{(1)},$$

for any j ($1 \leq j \leq r$).

In other words, the contextual meanings of \mathbf{f} and of \mathbf{t} are always the \mathfrak{T} -falsity and the \mathfrak{T} -truth, respectively.

On this basis, we put:

$$\mathbf{Hol}_{\mathfrak{T}}(\alpha) := \mathbf{Hol}_{\mathfrak{T}}(Level_1^\alpha),$$

for any formula α .

Since all gates are reversible, assigning a value $\mathbf{Hol}_{\mathfrak{T}}(Level_i^\alpha)$ to a particular $Level_i^\alpha$ of $STree^\alpha$ determines the value $\mathbf{Hol}_{\mathfrak{T}}(Level_j^\alpha)$ for any other level $Level_j^\alpha$. Consequently, $\mathbf{Hol}_{\mathfrak{T}}(Level_i^\alpha)$ determines the contextual meaning $\mathbf{Hol}_{\mathfrak{T}}^\alpha(\beta)$ for any subformula β of α .

Notice that any $\mathbf{Hol}_{\mathfrak{T}}(\alpha)$ represents a kind of autonomous semantic context that is not necessarily correlated with the meanings of other formulas. Generally we have:

$$\mathbf{Hol}_{\mathfrak{T}}^\gamma(\beta) \neq \mathbf{Hol}_{\mathfrak{T}}^\delta(\beta).$$

Thus, one and the same formula may receive different contextual meanings in different contexts (as, in fact, happens in the case of our normal use of natural languages).

DEFINITION 4.6 (Compositional holistic model). A holistic model $\mathbf{Hol}_{\mathfrak{T}}$ is called

- *compositional* iff $\mathbf{Hol}_{\mathfrak{T}}$ is a holistic map that is compositional with respect to all formulas α ;
- *perfectly compositional* iff $\mathbf{Hol}_{\mathfrak{T}}$ is a compositional model that satisfies the following condition for any formulas α, β and for any atomic formula \mathbf{q} (occurring in α and in β):

$$\mathbf{Hol}_{\mathfrak{T}}^\alpha(\mathbf{q}) = \mathbf{Hol}_{\mathfrak{T}}^\beta(\mathbf{q}).$$

Accordingly, models that are perfectly compositional are context-independent; while compositional models may be context-dependent. As expected, the *compositional quantum computational semantics*, that only refers to compositional models (or to perfectly compositional models), represents a special case of the holistic quantum computational semantics.

Consider now a formula α whose atomic complexity is n . By definition of model we have: $\mathbf{Hol}_{\mathfrak{T}}(\alpha) \in \mathfrak{D}(\mathcal{H}^{(n)})$. From an intuitive point of view, the qumix $\text{Red}_{[1, \dots, n]}^n(\mathbf{Hol}_{\mathfrak{T}}(\alpha))$ (which lives the space \mathbb{C}^2) can be regarded as a *generalized truth-value* of α (determined by the model $\mathbf{Hol}_{\mathfrak{T}}$). At the same time, the number $\mathbf{p}_{\mathfrak{T}}(\mathbf{Hol}(\alpha))$ represents the probability-value of α with respect to the truth-perspective \mathfrak{T} (determined by the model $\mathbf{Hol}_{\mathfrak{T}}$). Accordingly, our semantics can be described as a *two-level many valued semantics*, where for any choice of a model $\mathbf{Hol}_{\mathfrak{T}}$, any formula receives two correlated *semantic values*: a generalized truth-value (represented by a density operator of \mathbb{C}^2) and a probability-value (a real number in the interval $[0, 1]$).

To what extent do contextual meanings and gates (associated to the logical connectives) commute? In this respect the 1-ary connectives (\neg , $\sqrt{i\mathbf{d}}$ and $\sqrt{-\mathbf{i}}$) behave differently from the binary and the ternary connectives (\oplus and \top).

THEOREM 4.1. *Consider a model $\text{Hol}_{\mathfrak{T}}$.*

(1) *Let $\neg\beta$ be a subformula of γ . Then,*

$$\text{Hol}_{\mathfrak{T}}^{\gamma}(\neg\beta) = \mathfrak{D} \text{NOT}_{\mathfrak{T}}^{(\text{At}(\beta))}(\text{Hol}_{\mathfrak{T}}^{\gamma}(\beta)).$$

(2) *Let $\sqrt{\text{id}}\beta$ be a subformula of γ . Then,*

$$\text{Hol}_{\mathfrak{T}}^{\gamma}(\sqrt{\text{id}}\beta) = \mathfrak{D} \sqrt{\text{I}}_{\mathfrak{T}}^{(\text{At}(\beta))}(\text{Hol}_{\mathfrak{T}}^{\gamma}(\beta)).$$

(3) *Let $\sqrt{\neg}\beta$ be a subformula of γ . Then,*

$$\text{Hol}_{\mathfrak{T}}^{\gamma}(\sqrt{\neg}\beta) = \mathfrak{D} \sqrt{\text{NOT}}_{\mathfrak{T}}^{(\text{At}(\beta))}(\text{Hol}_{\mathfrak{T}}^{\gamma}(\beta)).$$

In other words, the contextual meaning of the negation of a formula β can be obtained by applying the appropriate negation-gate to the contextual meaning of β . In a similar way for the connectives $\sqrt{\text{id}}$ and $\sqrt{\neg}$.

Proof. By definition of syntactical tree, of \mathfrak{T} -gate tree, of holistic model and of contextual meaning. \square

Such a commutativity-situation breaks down in the case of the binary and ternary connectives (\uplus , Υ). As we have seen, the conjunction $\text{AND}_{\mathfrak{T}}^{(m,n)}$ has a characteristic holistic behavior. Generally, we have:

$$\text{AND}_{\mathfrak{T}}^{(m,n)}(\rho) = \mathfrak{D} \Upsilon_{\mathfrak{T}}^{(m,n,1)}(\rho \otimes \mathfrak{T} P_0^{(1)}) \neq \mathfrak{D} \Upsilon_{\mathfrak{T}}^{(m,n,1)}(\text{Red}_{[m,n,1]}^{(1)}(\rho) \otimes \text{Red}_{[m,n,1]}^{(2)}(\rho) \otimes \mathfrak{T} P_0^{(1)}).$$

Consequently, from a semantic point of view, we will generally obtain:

$$\text{Hol}_{\mathfrak{T}}^{\gamma}(\Upsilon(\alpha, \beta, \mathbf{f})) \neq \mathfrak{D} \Upsilon_{\mathfrak{T}}^{(\text{At}(\alpha), \text{At}(\beta), \text{At}(\mathbf{f}))}(\text{Hol}_{\mathfrak{T}}^{\gamma}(\alpha) \otimes \text{Hol}_{\mathfrak{T}}^{\gamma}(\beta) \otimes \text{Hol}_{\mathfrak{T}}^{\gamma}(\mathbf{f})).$$

A similar situation holds for the binary connective \uplus .

The connectives Υ and \uplus satisfy a weaker relation, described by the following theorem.

THEOREM 4.2. *Consider a model $\text{Hol}_{\mathfrak{T}}$.*

(1) *Let $\Upsilon(\alpha_1, \alpha_2, \alpha_3)$ be a subformula of γ . Thus, the syntactical tree of γ contains two levels having the following form:*

- $\text{Level}_{(i+1)}^{\gamma} = (\beta_{(i+1)_1}, \dots, \beta_{(i+1)_{k_1}}, \beta_{(i+1)_{k_2}}, \beta_{(i+1)_{k_3}}, \dots, \beta_{(i+1)_r})$,
where $\alpha_1 = \beta_{(i+1)_{k_1}}$, $\alpha_2 = \beta_{(i+1)_{k_2}}$, $\alpha_3 = \beta_{(i+1)_{k_3}}$.
- $\text{Level}_i^{\gamma} = (\beta_{i_1}, \dots, \beta_{i_j}, \dots, \beta_{i_s})$, where $\Upsilon(\alpha_1, \alpha_2, \alpha_3) = \beta_{i_j}$.

We have:

$$\begin{aligned} & \text{Hol}_{\mathfrak{T}}^{\gamma}(\Upsilon(\alpha_1, \alpha_2, \alpha_3)) \\ &= \mathfrak{D} \Upsilon_{\mathfrak{T}}^{(\text{At}(\alpha_1), \text{At}(\alpha_2), \text{At}(\alpha_3))} \left(\text{Red}_{[\text{At}(\beta_{(i+1)_1}), \dots, \text{At}(\beta_{(i+1)_r})]}^{(k_1, k_2, k_3)} (\text{Hol}_{\mathfrak{T}}(\text{Level}_{(i+1)}^{\gamma}(\gamma))) \right). \end{aligned}$$

(2) *A similar relation holds when $\alpha_1 \uplus \alpha_2$ is a subformula of γ .*

Proof. By definition of syntactical tree, of \mathfrak{T} -gate tree, of holistic model and of contextual meaning. \square

The holistic behavior of the connectives Υ and \uplus seem to reflect pretty well (at a semantic level) the holistic behavior of quantum circuits. As is well known, trying to separate the different branches “inside the box” of a given quantum computation generally has the effect of destroying the characteristic parallelism (and hence the efficiency) of the computation in question.

The following Lemma will play an important role in the development of the holistic semantics.

LEMMA 4.2. *Consider a formula γ and let η be a subformula of γ . For any model $\text{Hol}_{\mathfrak{T}}$ and for any formula β there exists a model $^*\text{Hol}_{\mathfrak{T}}$ such that,*

$$^*\text{Hol}_{\mathfrak{T}}^{\gamma \wedge \beta}(\eta) = \text{Hol}_{\mathfrak{T}}^{\gamma}(\eta).$$

Proof (Sketch). Consider two formulas γ and β and let $\text{Hol}_{\mathfrak{T}}$ be a model. If β is a subformula of γ the proof is trivial (since it is sufficient to take $^*\text{Hol}_{\mathfrak{T}}$ equal to $\text{Hol}_{\mathfrak{T}}$). Suppose that β is not a subformula of γ (while γ and β may have some common subformulas). Consider the syntactical tree of $\gamma \wedge \beta$, which includes (in its left part) the syntactical tree of γ (where $Level_1^{\gamma}$ appears at $Level_2^{\gamma \wedge \beta}$, while the top level of $STree^{\gamma}$ is supposed to be repeated until the Height h of $STree^{\gamma \wedge \beta}$ is reached). The model $\text{Hol}_{\mathfrak{T}}$ assigns a qumix $\text{Hol}_{\mathfrak{T}}(Level_i^{\gamma})$ to each level of $STree^{\gamma}$ (represented as a part of $STree^{\gamma \wedge \beta}$). Let us briefly write: ${}^{\gamma}\rho_{i+1} = \text{Hol}_{\mathfrak{T}}(Level_i^{\gamma})$. We transform $STree^{\gamma \wedge \beta}$ into a “hybrid” object $Hybr$ that is a sequence of sequences $Hybr_i$. Each $Hybr_i$ corresponds to $Level_i^{\gamma \wedge \beta}$ and is a sequence of objects that are either formulas or qumixes. Taking into account the fact the $\top(\gamma, \beta, \mathbf{f})$ and β are not subformulas of γ , we define the first two elements of $Hybr$ as follows:

$$Hybr_1 = (\top(\gamma, \beta, \mathbf{f})); \quad Hybr_2 = ({}^{\gamma}\rho_2, \beta, {}^{\mathfrak{T}}P_0^{(1)}).$$

Then, we proceed (step by step) by replacing the first occurrence in $STree^{\gamma \wedge \beta}$ of each formula θ that is also a subformula of γ with the qumix $\text{Hol}_{\mathfrak{T}}^{\gamma}(\theta)$. Suppose, for instance, that θ occurs for the first time at $Level_i^{\gamma \wedge \beta}$, and suppose that $\theta = \top(\xi_1, \xi_2, \xi_3)$. Then (by definition of syntactical tree), ξ_1 , ξ_2 and ξ_3 shall occur at $Level_{i+1}^{\gamma \wedge \beta}$. We define $Hybr_i$ and $Hybr_{i+1}$ in such a way that the following conditions are satisfied: a) in $Hybr_i$ the qumix $\text{Hol}_{\mathfrak{T}}^{\gamma}(\theta)$ occurs in place of the formula θ (occurring at $Level_i^{\gamma \wedge \beta}$); b) in $Hybr_{i+1}$ the qumix $[{}^{\mathfrak{T}}\mathbf{T}_{\mathfrak{T}}^{(\text{At}(\xi_1), \text{At}(\xi_2), \text{At}(\xi_3))}]^{-1}(\text{Hol}_{\mathfrak{T}}^{\gamma}(\theta))$ occurs in place of the subsequence (ξ_1, ξ_2, ξ_3) (occurring at $Level_{i+1}^{\gamma \wedge \beta}$). We proceed in a similar way for all possible linguistic forms of θ . When we finally reach the top level $Level_h^{\gamma \wedge \beta}$, the corresponding $Hybr_h$ will have the following form:

$$Hybr_h = ({}^{\gamma}\rho_h, Ob_1, \dots, Ob_t, {}^{\mathfrak{T}}P_0^{(1)}),$$

where each Ob_j is either a qumix or an atomic formula \mathbf{q} that does not occur in γ . Now, we replace in $Hybr_h$ each “surviving” formula \mathbf{q} with the qumix $\text{Hol}_{\mathfrak{T}}(\mathbf{q})$ (which lives in \mathbb{C}^2). This operation destroys the “hybrid” form of $Hybr_h$, which is now transformed into a homogeneous sequence of qumixes:

$${}^{\mathfrak{D}}Hybr_h = ({}^{\gamma}\rho_h, {}^{\mathfrak{D}}Ob_1, \dots, {}^{\mathfrak{D}}Ob_t, {}^{\mathfrak{T}}P_0^{(1)}), \quad \text{where :}$$

$${}^{\mathfrak{D}}Ob_j = \begin{cases} Ob_j, & \text{if } Ob_j \text{ is a qumix;} \\ \text{Hol}_{\mathfrak{T}}(\mathbf{q}), & \text{if } Ob_j = \mathbf{q}. \end{cases}$$

On this basis, we transform the whole $Hybr$ into a sequence of qumix-sequences ${}^{\mathfrak{D}}Hybr_i$. Let us first refer to $Hybr_{h-1}$, which may contain formulas that are not subformulas of γ . Suppose, for instance, that the first formula occurring in $Hybr_{h-1}$ is

$$\beta_{(h-1)_j} = \top(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3).$$

Since $\beta_{(h-1)_j}$ is not a subformula of γ , ${}^{\mathfrak{D}}Hybr_h$ shall contain three separate qumixes $\mathbf{q}_1\rho$, $\mathbf{q}_2\rho$, $\mathbf{q}_3\rho$ (corresponding to the atom-sequence $(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)$ occurring in the right part of $STree^{\gamma \wedge \beta}$). On this basis, we replace the formula $\top(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)$ with the qumix ${}^{\mathfrak{D}}\mathbf{T}_{\mathfrak{T}}^{(1,1,1)}(\mathbf{q}_1\rho \otimes \mathbf{q}_2\rho \otimes \mathbf{q}_3\rho)$ in $Hybr_{h-1}$ and in all other $Hybr_i$ where $\top(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)$ possibly appears.

Then, we proceed step by step by applying the same procedure to all formulas β_{i_j} occurring in $Hybr_i$, for any i ($1 \leq i < h$). At the end of the procedure, each $Hybr_i$ ($1 < i \leq h$) has been transformed into a sequence of qumixes

$$\mathfrak{D}Hybr_i = (\gamma\rho_i, \rho_{i_1}, \dots, \rho_{i_r}, \mathfrak{T}P_0^{(1)}),$$

where any qumix is naturally associated to a segment of $Level_i^{\gamma \wedge \beta}$.

We define now the map $*\text{Hol}_{\mathfrak{T}}$ in the following way:

- $*\text{Hol}_{\mathfrak{T}}(Level_i^{\gamma \wedge \beta}) = \gamma\rho_i \otimes \rho_{i_1} \otimes \dots \otimes \rho_{i_r} \otimes \mathfrak{T}P_0^{(1)}$, if $1 < i \leq h$;
- $*\text{Hol}_{\mathfrak{T}}(Level_i^{\gamma \wedge \beta}) = \mathfrak{D}\mathfrak{T}^{(\text{At}(\gamma), \text{At}(\beta), 1)}(*\text{Hol}_{\mathfrak{T}}(Level_2^{\gamma \wedge \beta}))$, if $i = 1$.

We have:

- (I) by construction, $*\text{Hol}_{\mathfrak{T}}(Level_i^{\gamma \wedge \beta})$ is a qumix of $\mathcal{H}^{\gamma \wedge \beta}$. Hence, $*\text{Hol}_{\mathfrak{T}}$ is a holistic map for $\gamma \wedge \beta$;
- (II) $*\text{Hol}_{\mathfrak{T}}$ is normal for $\gamma \wedge \beta$, by the normality of $\text{Hol}_{\mathfrak{T}}$ and because different occurrences in $Hybr$ of a formula that is not a subformula of γ have been replaced by the same qumix;
- (III) by construction, $*\text{Hol}_{\mathfrak{T}}$ preserves the logical form of all subformulas of $\gamma \wedge \beta$. Accordingly, $*\text{Hol}_{\mathfrak{T}}(Level_i^{\gamma \wedge \beta}) = \mathfrak{D}G_{\mathfrak{T}(i)}(*\text{Hol}_{\mathfrak{T}}(Level_{i+1}^{\gamma \wedge \beta}))$, for any i such that $1 \leq i < h$, where $(\mathfrak{D}G_{\mathfrak{T}(h-1)}, \dots, \mathfrak{D}G_{\mathfrak{T}(1)})$ is the \mathfrak{T} -gate tree of $\gamma \wedge \beta$. Furthermore, the sentences \mathbf{f} and \mathbf{t} have (trivially) the “right” contextual meanings. Hence, $*\text{Hol}_{\mathfrak{T}}$ is a model for $\gamma \wedge \beta$;
- (IV) by construction, for any η that is a subformula of γ :

$$*\text{Hol}_{\mathfrak{T}}^{\gamma \wedge \beta}(\eta) = \text{Hol}_{\mathfrak{T}}^{\gamma}(\eta). \quad \square$$

Now the concepts of *truth*, *validity*, *logical consequence* and *logical equivalence* can be defined in terms of the probability-function $\mathbf{p}_{\mathfrak{T}}$ and of the preorder $\preceq_{\mathfrak{T}}$.

DEFINITION 4.7 (Truth). A formula α is called *true* with respect to a model $\text{Hol}_{\mathfrak{T}}$ (abbreviated as $\models_{\text{Hol}_{\mathfrak{T}}} \alpha$) iff $\mathbf{p}_{\mathfrak{T}}(\text{Hol}_{\mathfrak{T}}(\alpha)) = 1$.

DEFINITION 4.8 (Validity).

- 1) α is called *\mathfrak{T} -valid* ($\models_{\mathfrak{T}} \alpha$) iff for any model $\text{Hol}_{\mathfrak{T}}$, $\models_{\text{Hol}_{\mathfrak{T}}} \alpha$.
- 2) α is called *valid* ($\models \alpha$) iff for any truth-perspective \mathfrak{T} , $\models_{\mathfrak{T}} \alpha$.

DEFINITION 4.9 (Logical consequence).

- 1) β is called a *\mathfrak{T} -logical consequence* of α ($\alpha \models_{\mathfrak{T}} \beta$) iff for any formula γ such that α and β are subformulas of γ and for any model $\text{Hol}_{\mathfrak{T}}$,

$$\text{Hol}_{\mathfrak{T}}^{\gamma}(\alpha) \preceq_{\mathfrak{T}} \text{Hol}_{\mathfrak{T}}^{\gamma}(\beta).$$

- 2) β is called a *logical consequence* of α ($\alpha \models \beta$) iff for any truth-perspective \mathfrak{T} , $\alpha \models_{\mathfrak{T}} \beta$.

When $\alpha \models_{\mathfrak{T}} \beta$, we say that β is a *canonical logical consequence* of α .

DEFINITION 4.10 (Logical equivalence). α and β are logically equivalent ($\alpha \equiv \beta$) iff $\alpha \models \beta$ and $\beta \models \alpha$.

The concept of logical consequence turns out to be invariant with respect to truth-perspective changes.

LEMMA 4.3. ([2]) $\alpha \models \beta$ iff $\alpha \models_{\mathfrak{T}} \beta$ iff there is a truth-perspective \mathfrak{T} such that $\alpha \models_{\mathfrak{T}} \beta$.

Although the holistic semantics is strongly context-dependent, one can prove that the logical consequence-relation is reflexive and transitive.

THEOREM 4.3.

- (1) $\alpha \vDash \alpha$;
- (2) $\alpha \vDash \beta$ and $\beta \vDash \delta \implies \alpha \vDash \delta$.

Proof.

(1) Straightforward.

(2) Assume the hypothesis and suppose, by contradiction, that there exist a model $\text{Hol}_{\mathfrak{I}}$ and a formula γ , where α and δ occur as subformulas, such that: $\text{Hol}_{\mathfrak{I}}^{\gamma}(\alpha) \not\leq_{\mathfrak{I}} \text{Hol}_{\mathfrak{I}}^{\gamma}(\delta)$. Consider the formula $\gamma \wedge \beta$. By Lemma 4.2 there exists a model $^*\text{Hol}_{\mathfrak{I}}$ such that for any η that is a subformula of γ : $^*\text{Hol}_{\mathfrak{I}}^{\gamma \wedge \beta}(\eta) = \text{Hol}_{\mathfrak{I}}^{\gamma}(\eta)$. Thus, we have:

$$^*\text{Hol}_{\mathfrak{I}}^{\gamma \wedge \beta}(\alpha) = \text{Hol}_{\mathfrak{I}}^{\gamma}(\alpha)$$

and

$$^*\text{Hol}_{\mathfrak{I}}^{\gamma \wedge \beta}(\delta) = \text{Hol}_{\mathfrak{I}}^{\gamma}(\delta).$$

Since we have assumed (by contradiction) that $\text{Hol}_{\mathfrak{I}}^{\gamma}(\alpha) \not\leq_{\mathfrak{I}} \text{Hol}_{\mathfrak{I}}^{\gamma}(\delta)$, we obtain: $^*\text{Hol}_{\mathfrak{I}}^{\gamma \wedge \beta}(\alpha) \not\leq_{\mathfrak{I}} ^*\text{Hol}_{\mathfrak{I}}^{\gamma \wedge \beta}(\delta)$, against the hypothesis and the transitivity of $\leq_{\mathfrak{I}}$, which imply:

$$^*\text{Hol}_{\mathfrak{I}}^{\gamma \wedge \beta}(\alpha) \leq_{\mathfrak{I}} ^*\text{Hol}_{\mathfrak{I}}^{\gamma \wedge \beta}(\beta);$$

$$^*\text{Hol}_{\mathfrak{I}}^{\gamma \wedge \beta}(\beta) \leq_{\mathfrak{I}} ^*\text{Hol}_{\mathfrak{I}}^{\gamma \wedge \beta}(\delta);$$

$$^*\text{Hol}_{\mathfrak{I}}^{\gamma \wedge \beta}(\alpha) \leq_{\mathfrak{I}} ^*\text{Hol}_{\mathfrak{I}}^{\gamma \wedge \beta}(\delta). \quad \square$$

The concept of logical consequence, defined in this semantics, characterizes a special form of quantum computational logic (formalized in the language \mathcal{L}) that is termed *holistic quantum computational logic* (**HQCL**). One can easily show that **HQCL** includes, as a particular fragment, classical sentential logic (representing also an adequate description of classical circuits).

Consider the sublanguage \mathcal{L}^C of \mathcal{L} , whose formulas are the Boolean formulas of \mathcal{L} .

DEFINITION 4.11 (Classical quantum computational model). A *classical quantum computational model* is a model $\text{Hol}_{\mathfrak{I}}$ that satisfies the following conditions:

- (1) \mathfrak{I} is the canonical truth-perspective \mathbf{I} .
- (2) $\text{Hol}_{\mathfrak{I}}$ is only defined for \mathcal{L}^C -formulas.
- (3) For any formula α of \mathcal{L}^C , $\text{Hol}_{\mathfrak{I}}$ assigns to the top level of the syntactical tree of α a (canonical) register (living in the semantic space of α).

One immediately obtains that any classical quantum computational model assigns to any Boolean formula of \mathcal{L} a (canonical) register (living in its semantic space).

We can now define a consequence-relation that concerns the Boolean language \mathcal{L}^C .

DEFINITION 4.12 (Classical quantum computational consequence). A formula β of \mathcal{L}^C is called a *classical quantum computational consequence* of a formula α of \mathcal{L}^C ($\alpha \vDash_{CQC} \beta$) iff for any formula γ of \mathcal{L}^C such that α and β are subformulas of γ and for any classical quantum computational model $\text{Hol}_{\mathbf{I}}$,

$$\text{Hol}_{\mathbf{I}}^{\gamma}(\alpha) \leq_{\mathbf{I}} \text{Hol}_{\mathbf{I}}^{\gamma}(\beta).$$

LEMMA 4.4. *For any formulas α and β of \mathcal{L}^C , $\alpha \vDash_{CQC} \beta$ iff β is a logical consequence of α according to classical sentential logic.*

Proof. Straightforward. □

5. Logical arguments

Which logical arguments are valid or are possibly violated in the logic **HQCL**? The following theorems give some answers to this question. By Lemma 4.3 it will be sufficient to refer to the canonical logical consequence relation and to canonical models. Accordingly, we will write: \mathbf{p} , \preceq , \mathbf{Hol} and \vDash (instead of \mathbf{p}_I , \preceq_I , \mathbf{Hol}_I and \vDash_I).

Theorem 5.1 sums up some basic arguments that hold for the quantum computational Boolean connectives.

THEOREM 5.1.

- (1) $\alpha \wedge \beta \vDash \alpha$; $\alpha \wedge \beta \vDash \beta$
- (2) $\alpha \vDash \beta \implies \alpha \wedge \delta \vDash \beta$
- (3) $\neg\neg\alpha \equiv \alpha$
- (4) $\alpha \vDash \beta \implies \neg\beta \vDash \neg\alpha$
- (5) $\mathbf{f} \vDash \beta$; $\beta \vDash \mathbf{t}$

Proof.

- (1) $\alpha \wedge \beta \vDash \alpha$; $\alpha \wedge \beta \vDash \beta$.

Let α and $\alpha \wedge \beta$ ($= \top(\alpha, \beta, \mathbf{f})$) be subformulas of γ . Suppose that α , β , \mathbf{f} occur respectively at the positions k_1 , k_2 , k_3 of $Level_{i+1}^\gamma$ (in the syntactical tree of γ), while $\top(\alpha, \beta, \mathbf{f})$ occurs at $Level_i^\gamma$. By Theorem 4.2(1), for any \mathbf{Hol} we have:

$$\mathbf{Hol}^\gamma(\top(\alpha, \beta, \mathbf{f})) = \mathfrak{D}_{\top(\text{At}(\alpha), \text{At}(\beta), \text{At}(\mathbf{f}))}(\text{Red}_{[1, \dots, r]}^{(k_1, k_2, k_3)}(\mathbf{Hol}(Level_{i+1}^\gamma)))$$

(where r is the number of formulas occurring at $Level_{i+1}^\gamma$). Hence, by definition of contextual meaning and by Theorem 3.2:

$$\mathbf{Hol}^\gamma(\top(\alpha, \beta, \mathbf{f})) \preceq \mathbf{Hol}^\gamma(\alpha).$$

In a similar way one can prove that $\alpha \wedge \beta \vDash \beta$.

- (2) $\alpha \vDash \beta \implies \alpha \wedge \delta \vDash \beta$.

Assume the hypothesis and let $\alpha \wedge \delta$, β be subformulas of γ . Then α and δ also are subformulas of γ . By hypothesis, for any \mathbf{Hol} : $\mathbf{Hol}^\gamma(\alpha) \preceq \mathbf{Hol}^\gamma(\beta)$. By (1): $\mathbf{Hol}^\gamma(\alpha \wedge \delta) \preceq \mathbf{Hol}^\gamma(\alpha)$. Hence, by transitivity of \preceq : $\mathbf{Hol}^\gamma(\alpha \wedge \delta) \preceq \mathbf{Hol}^\gamma(\beta)$.

- (3) $\neg\neg\alpha \equiv \alpha$.

Let $\neg\neg\alpha$ and α be subformulas of γ . By Theorem 4.1 (1) and by the double-negation principle for the gate $\mathfrak{D}\text{NOT}^{(n)}$, we obtain for any \mathbf{Hol} :

$$\mathbf{Hol}^\gamma(\neg\neg\alpha) = \mathfrak{D}\text{NOT}^{(\text{At}(\alpha))} \mathfrak{D}\text{NOT}^{(\text{At}(\alpha))} \mathbf{Hol}^\gamma(\alpha) = \mathbf{Hol}^\gamma(\alpha).$$

- (4) $\alpha \vDash \beta \implies \neg\beta \vDash \neg\alpha$.

Assume the hypothesis and let $\neg\beta$, $\neg\alpha$ be subformulas of γ . Then α and β also are subformulas of γ . By hypothesis, for any \mathbf{Hol} , $\mathbf{p}(\mathbf{Hol}^\gamma(\alpha)) \leq \mathbf{p}(\mathbf{Hol}^\gamma(\beta))$. Hence, $1 - \mathbf{p}(\mathbf{Hol}^\gamma(\beta)) \leq 1 - \mathbf{p}(\mathbf{Hol}^\gamma(\alpha))$. Since for any $\rho \in \mathfrak{D}(\mathcal{H}^{(n)})$, $\mathbf{p}(\mathfrak{D}\text{NOT}^{(n)}\rho) = 1 - \mathbf{p}(\rho)$, we obtain:

$$\mathfrak{D}\text{NOT}^{(\text{At}(\beta))} \mathbf{Hol}^\gamma(\beta) \preceq \mathfrak{D}\text{NOT}^{(\text{At}(\alpha))} \mathbf{Hol}^\gamma(\alpha).$$

Whence, by Theorem 4.1 (1), $\text{Hol}^\gamma(\neg\beta) \preceq \text{Hol}^\gamma(\neg\alpha)$.

$$(5) \mathbf{f} \models \beta; \beta \models \mathbf{t}.$$

Let β and \mathbf{f} be subformulas of γ . By definition of holistic model we have: $\mathbf{p}(\text{Hol}^\gamma(\mathbf{f})) = \mathbf{p}(P_0^{(1)}) = 0$, for any Hol . Hence, $\text{Hol}^\gamma(\mathbf{f}) \preceq \text{Hol}^\gamma(\beta)$. In a similar way one proves that $\beta \models \mathbf{t}$. \square

The dual forms of 5.1(1) and of 5.1(2) hold for the connective \vee .

The following theorem sums up some significant classical arguments that are not valid for the quantum computational Boolean connectives.

THEOREM 5.2.

- (1) $\alpha \not\models \alpha \wedge \alpha$
- (2) $\alpha \wedge \beta \not\models \beta \wedge \alpha$
- (3) $\alpha \wedge (\beta \wedge \delta) \not\models (\alpha \wedge \beta) \wedge \delta$
- (4) $(\alpha \wedge \beta) \wedge \delta \not\models \alpha \wedge (\beta \wedge \delta)$
- (5) $\alpha \wedge (\beta \vee \delta) \not\models (\alpha \wedge \beta) \vee (\alpha \wedge \delta)$
- (6) $(\alpha \wedge \beta) \vee (\alpha \wedge \delta) \not\models \alpha \wedge (\beta \vee \delta)$
- (7) $\delta \models \alpha$ and $\delta \models \beta \not\Rightarrow \delta \models \alpha \wedge \beta$
- (8) $\alpha \wedge \neg\alpha \not\models \beta$
- (9) $\alpha \uplus \beta \not\models \beta \uplus \alpha$
- (10) $\alpha \uplus \beta \not\models \alpha \vee \beta; \quad \alpha \uplus \beta \not\models \neg\alpha \vee \neg\beta$

Proof. In the following counterexamples α , β and δ will always represent atomic formulas.

$$(1) \alpha \not\models \alpha \wedge \alpha$$

Take $\gamma = \alpha \wedge \alpha$ and consider a model Hol such that

$$\text{Hol}(\gamma) = \mathfrak{D}\mathbb{T}^{(1,1,1)} \left(\frac{1}{2}\mathbb{I} \otimes \frac{1}{2}\mathbb{I} \otimes P_0^{(1)} \right).$$

We have: $\mathbf{p}(\text{Hol}^\gamma(\alpha)) = \frac{1}{2} > \mathbf{p}(\text{Hol}^\gamma(\alpha \wedge \alpha)) = \frac{1}{4}$.

$$(2) \alpha \wedge \beta \not\models \beta \wedge \alpha.$$

Take $\gamma = (\alpha \wedge \beta) \wedge (\beta \wedge \alpha)$. Consider a model Hol such that

$$\text{Hol}(\gamma) = \mathfrak{D}\mathbb{T}^{(3,3,1)} \left[\mathfrak{D}\mathbb{T}^{(1,1,1)} \left(\frac{1}{2}\mathbb{I} \otimes \frac{1}{2}\mathbb{I} \otimes P_0^{(1)} \right) \otimes \mathfrak{D}\mathbb{T}^{(1,1,1)} \left(P_{\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)} \otimes P_0^{(1)} \otimes P_0^{(1)} \right) \right].$$

We have: $\mathbf{p}(\text{Hol}^\gamma(\alpha \wedge \beta)) = \frac{1}{4} > \mathbf{p}(\text{Hol}^\gamma(\beta \wedge \alpha)) = 0$.

$$(3) \alpha \wedge (\beta \wedge \delta) \not\models (\alpha \wedge \beta) \wedge \delta.$$

Take $\gamma = (\alpha \wedge (\beta \wedge \delta)) \wedge ((\alpha \wedge \beta) \wedge \delta)$ and consider a model Hol such that

$$\begin{aligned} \text{Hol}(\gamma) = \mathfrak{D}\mathbb{T}^{(5,5,1)} \left[\mathfrak{D}\mathbb{T}^{(1,3,1)} \left(\frac{1}{2}\mathbb{I} \otimes \mathfrak{D}\mathbb{T}^{(1,1,1)} \left(\frac{1}{2}\mathbb{I} \otimes \frac{1}{2}\mathbb{I} \otimes P_0^{(1)} \right) \otimes P_0^{(1)} \right) \right. \\ \left. \otimes \mathfrak{D}\mathbb{T}^{(3,1,1)} \left(\mathfrak{D}\mathbb{T}^{(1,1,1)} \otimes \mathbb{I}^{(2)} \left(P_{\frac{1}{\sqrt{2}}(|01010\rangle - |10000\rangle)} \right) \right) \otimes P_0^{(1)} \right]. \end{aligned}$$

We have: $\mathbf{p}(\text{Hol}^\gamma(\alpha \wedge (\beta \wedge \delta))) = \frac{1}{8} > \mathbf{p}(\text{Hol}^\gamma((\alpha \wedge \beta) \wedge \delta)) = 0$.

$$(4) (\alpha \wedge \beta) \wedge \delta \not\models \alpha \wedge (\beta \wedge \delta).$$

Similar to (3).

(5) $\alpha \wedge (\beta \vee \delta) \not\models (\alpha \wedge \beta) \vee (\alpha \wedge \delta)$.

Take $\gamma = (\alpha \wedge (\beta \vee \delta)) \wedge ((\alpha \wedge \beta) \vee (\alpha \wedge \delta))$. Consider a model Ho1 such that

$$\begin{aligned} \text{Ho1}(\gamma) &= \mathfrak{D}_{\mathbf{T}(5,7,1)} \left[\mathfrak{D}_{\mathbf{T}(1,3,1)} \left(\frac{1}{2}\mathbf{I} \otimes \mathfrak{D}_{\text{NOT}^{(3)}} \mathfrak{D}_{\mathbf{T}(1,1,1)} \left(P_{\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)} \otimes P_0^{(1)} \right) \otimes P_0^{(1)} \right) \right. \\ &\quad \otimes \mathfrak{D}_{\text{NOT}^{(7)}} \mathfrak{D}_{\mathbf{T}(3,3,1)} \left(\mathfrak{D}_{\text{NOT}^{(3)}} \mathfrak{D}_{\mathbf{T}(1,1,1)} \left(\frac{1}{2}\mathbf{I} \otimes \frac{1}{2}\mathbf{I} \otimes P_0^{(1)} \right) \right) \\ &\quad \left. \otimes \mathfrak{D}_{\text{NOT}^{(3)}} \mathfrak{D}_{\mathbf{T}(1,1,1)} \left(\frac{1}{2}\mathbf{I} \otimes \frac{1}{2}\mathbf{I} \otimes P_0^{(1)} \right) \otimes P_0^{(1)} \right] \otimes P_0^{(1)}. \end{aligned}$$

We have: $\text{p}(\text{Ho1}^\gamma(\alpha \wedge (\beta \vee \delta))) = \frac{1}{2} > \text{p}(\text{Ho1}^\gamma((\alpha \wedge \beta) \vee (\alpha \wedge \delta))) = \frac{7}{16}$.

(6) $(\alpha \wedge \beta) \vee (\alpha \wedge \delta) \not\models \alpha \wedge (\beta \vee \delta)$.

Take $\gamma = (\alpha \wedge (\beta \vee \delta)) \wedge ((\alpha \wedge \beta) \vee (\alpha \wedge \delta))$.

Consider a model Ho1 such that

$$\begin{aligned} \text{Ho1}(\gamma) &= \mathfrak{D}_{\mathbf{T}(5,7,1)} \left[\mathfrak{D}_{\mathbf{T}(1,3,1)} \left(\frac{1}{2}\mathbf{I} \otimes \mathfrak{D}_{\text{NOT}^{(3)}} \mathfrak{D}_{\mathbf{T}(1,1,1)} \left(\frac{1}{2}\mathbf{I} \otimes \frac{1}{2}\mathbf{I} \otimes P_0^{(1)} \right) \otimes P_0^{(1)} \right) \right. \\ &\quad \otimes \mathfrak{D}_{\text{NOT}^{(7)}} \mathfrak{D}_{\mathbf{T}(3,3,1)} \left(\mathfrak{D}_{\text{NOT}^{(3)}} \mathfrak{D}_{\mathbf{T}(1,1,1)} \left(\frac{1}{2}\mathbf{I} \otimes \frac{1}{2}\mathbf{I} \otimes P_0^{(1)} \right) \right) \\ &\quad \left. \otimes \mathfrak{D}_{\text{NOT}^{(3)}} \mathfrak{D}_{\mathbf{T}(1,1,1)} \left(\frac{1}{2}\mathbf{I} \otimes \frac{1}{2}\mathbf{I} \otimes P_0^{(1)} \right) \otimes P_0^{(1)} \right] \otimes P_0^{(1)}. \end{aligned}$$

We have: $\text{p}(\text{Ho1}^\gamma((\alpha \wedge \beta) \vee (\alpha \wedge \delta))) = \frac{7}{16} > \text{p}(\text{Ho1}^\gamma(\alpha \wedge (\beta \vee \delta))) = \frac{3}{8}$.

(7) $\delta \models \alpha$ and $\delta \models \beta \not\Rightarrow \delta \models \alpha \wedge \beta$.

Take $\gamma = (\alpha \wedge \beta) \wedge \delta$. Consider a model Ho1 such that

$$\text{Ho1}(\gamma) = \mathfrak{D}_{\mathbf{T}(3,1,1)} \left(\mathfrak{D}_{\mathbf{T}(1,1,1)} \left(\frac{1}{2}\mathbf{I} \otimes \frac{1}{2}\mathbf{I} \otimes P_0^{(1)} \right) \otimes \frac{1}{2}\mathbf{I} \otimes P_0^{(1)} \right).$$

We have: $\text{p}(\text{Ho1}^\gamma(\alpha)) = \text{p}(\text{Ho1}^\gamma(\beta)) = \text{p}(\text{Ho1}^\gamma(\delta)) = \frac{1}{2} > \text{p}(\text{Ho1}^\gamma(\alpha \wedge \beta)) = \frac{1}{4}$.

(8) $\alpha \wedge \neg\alpha \not\models \beta$.

Take $\gamma = (\alpha \wedge \neg\alpha) \wedge \beta$. Consider a model Ho1 such that

$$\text{Ho1}(\gamma) = \mathfrak{D}_{\mathbf{T}(3,1,1)} \left(\mathfrak{D}_{\mathbf{T}(1,1,1)} \left(\frac{1}{2}\mathbf{I} \otimes \frac{1}{2}\mathbf{I} \otimes P_0^{(1)} \right) \otimes P_0^{(1)} \otimes P_0^{(1)} \right).$$

We have: $\text{p}(\text{Ho1}^\gamma(\alpha \wedge \neg\alpha)) = \frac{1}{4}$ and $\text{p}(\text{Ho1}^\gamma(\beta)) = 0$.

(9) $\alpha \uplus \beta \not\models \beta \uplus \alpha$.

Take $\gamma = (\alpha \uplus \beta) \wedge (\beta \uplus \alpha)$. Consider a model Ho1 such that

$$\text{Ho1}(\gamma) = \mathfrak{D}_{\mathbf{T}(2,2,1)} \left[\mathfrak{D}_{\text{XOR}^{(1,1)}} P_{\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)} \otimes \mathfrak{D}_{\text{XOR}^{(1,1)}} \left(\frac{1}{2}\mathbf{I} \otimes \frac{1}{2}\mathbf{I} \right) \otimes P_0^{(1)} \right].$$

We have: $\text{p}(\text{Ho1}^\gamma(\alpha \uplus \beta)) = 1 > \text{p}(\text{Ho1}^\gamma(\beta \uplus \alpha)) = \frac{1}{2}$.

(10) $\alpha \uplus \beta \not\models \alpha \vee \beta$; $\alpha \uplus \beta \not\models \neg\alpha \vee \neg\beta$.

Take $\gamma = (\alpha \uplus \beta) \wedge (\alpha \vee \beta)$. Consider a model Ho1 such that

$$\begin{aligned} \text{Ho1}(\gamma) &= \mathfrak{D}_{\mathbf{T}(2,3,1)} \left[\mathfrak{D}_{\text{XOR}^{(1,1)}} P_{\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)} \otimes \mathfrak{D}_{\text{NOT}^{(3)}} \mathfrak{D}_{\mathbf{T}(1,1,1)} \left(\text{NOT}^{(1)} \otimes \text{NOT}^{(1)} \otimes \mathbf{I}^{(1)} \right) \right. \\ &\quad \left. \cdot \left(\frac{1}{2}\mathbf{I} \otimes \frac{1}{2}\mathbf{I} \otimes P_0^{(1)} \right) \otimes P_0^{(1)} \right]. \end{aligned}$$

We have: $\text{p}(\text{Ho1}^\gamma(\alpha \uplus \beta)) = 1 > \text{p}(\text{Ho1}^\gamma(\alpha \vee \beta)) = \frac{3}{4}$.

In a similar way one proves that $\alpha \uplus \beta \not\models \neg\alpha \vee \neg\beta$. □

The dual forms of 5.2(1)–5.1(8) hold for the connective \vee .

The following theorem sums up some basic arguments that hold for the genuine quantum computational connectives.

THEOREM 5.3.

- (1) $\sqrt{\text{id}} \sqrt{\text{id}} \alpha \equiv \alpha$
- (2) $\sqrt{\text{id}} \mathbf{f} \equiv \sqrt{\text{id}} \mathbf{t}$
- (3) $\neg \sqrt{\text{id}} \mathbf{f} \equiv \sqrt{\text{id}} \mathbf{f}$; $\neg \sqrt{\text{id}} \mathbf{t} \equiv \sqrt{\text{id}} \mathbf{t}$
- (4) $\sqrt{\text{id}} (\alpha \wedge \beta) \equiv \sqrt{\text{id}} \mathbf{f}$
- (5) $\sqrt{\neg} \sqrt{\neg} \alpha \equiv \neg \alpha$
- (6) $\sqrt{\neg} \mathbf{f} \equiv \sqrt{\neg} \mathbf{t}$
- (7) $\neg \sqrt{\neg} \mathbf{f} \equiv \sqrt{\neg} \mathbf{f}$; $\neg \sqrt{\neg} \mathbf{t} \equiv \sqrt{\neg} \mathbf{t}$
- (8) $\neg \sqrt{\neg} \alpha \equiv \sqrt{\neg} \neg \alpha$
- (9) $\sqrt{\neg} (\alpha \wedge \beta) \equiv \sqrt{\neg} \mathbf{f}$
- (10) $\sqrt{\text{id}} \sqrt{\neg} \alpha \equiv \sqrt{\text{id}} \alpha$
- (11) $\sqrt{\neg} \sqrt{\text{id}} \alpha \equiv \neg \sqrt{\neg} \alpha$
- (12) $\sqrt{\text{id}} \sqrt{\neg} (\alpha \wedge \beta) \equiv \sqrt{\neg} \mathbf{f}$
- (13) $\sqrt{\neg} \sqrt{\text{id}} (\alpha \wedge \beta) \equiv \sqrt{\neg} \mathbf{f}$

Proof.

- (1) $\sqrt{\text{id}} \sqrt{\text{id}} \alpha \equiv \alpha$.

Let α and $\sqrt{\text{id}} \sqrt{\text{id}} \alpha$ be subformulas of γ . Since for any $\rho \in \mathfrak{D}(\mathcal{H}^{(n)})$, $\mathfrak{D} \sqrt{\text{id}}^{(n)} \mathfrak{D} \sqrt{\text{id}}^{(n)} \rho = \rho$, by Theorem 4.1(2) we obtain: $\text{Ho1}^\gamma(\sqrt{\text{id}} \sqrt{\text{id}} \alpha) = \text{Ho1}^\gamma(\alpha)$.

- (2) $\sqrt{\text{id}} \mathbf{f} \equiv \sqrt{\text{id}} \mathbf{t}$

By definition of model, by Theorem 4.1(2) and because $\text{p} \left(\mathfrak{D} \sqrt{\text{id}}^{(1)} P_0^{(1)} \right) = \frac{1}{2} = \text{p} \left(\mathfrak{D} \sqrt{\text{id}}^{(1)} P_1^{(1)} \right)$.

- (3) $\neg \sqrt{\text{id}} \mathbf{f} \equiv \sqrt{\text{id}} \mathbf{f}$; $\neg \sqrt{\text{id}} \mathbf{t} \equiv \sqrt{\text{id}} \mathbf{t}$

Let $\neg \sqrt{\text{id}} \mathbf{f}$ and $\sqrt{\text{id}} \mathbf{f}$ be subformulas of γ . By definition of model and by Theorem 4.1(1,2) we have:

$$\text{p} \left(\text{Ho1}^\gamma(\neg \sqrt{\text{id}} \mathbf{f}) \right) = \text{p} \left(\mathfrak{D} \text{NOT}^{(1)} \mathfrak{D} \sqrt{\text{id}}^{(1)} P_0^{(1)} \right) = \frac{1}{2} = \text{p} \left(\mathfrak{D} \sqrt{\text{id}}^{(1)} P_0^{(1)} \right) = \text{Ho1}^\gamma \left(\sqrt{\text{id}} \mathbf{f} \right).$$

In a similar way one proves that $\neg \sqrt{\text{id}} \mathbf{t} \equiv \sqrt{\text{id}} \mathbf{t}$.

- (4) $\sqrt{\text{id}} (\alpha \wedge \beta) \equiv \sqrt{\text{id}} \mathbf{f}$.

Let $\sqrt{\text{id}} (\alpha \wedge \beta)$ and $\sqrt{\text{id}} \mathbf{f}$ be subformulas of γ . Let $\text{At}(\alpha) = m$ and $\text{At}(\beta) = n$. Suppose that in the syntactical tree of γ the subformulas α and β occur, respectively, at the positions k_1 and k_2 of Level_i^γ (consisting of r formulas). Consider a model Ho1 and let $\rho = \text{Red}_{[1, \dots, r]}^{(k_1, k_2)}(\text{Ho1}(\text{Level}_i^\gamma))$.

By Theorem 4.2(1), we have: $\text{Ho1}^\gamma(\alpha \wedge \beta) = \mathfrak{D} \mathbb{T}^{(m, n, 1)}(\rho \otimes P_0^{(1)})$. Then, by definition of p and by Lemma 3.1 we obtain:

$$\begin{aligned} \text{p} \left(\text{Ho1}^\gamma(\sqrt{\text{id}} (\alpha \wedge \beta)) \right) &= \text{tr} \left[P_1^{(m+n+1)} \mathfrak{D} \sqrt{\text{id}}^{(m+n+1)} \mathfrak{D} \mathbb{T}^{(m, n, 1)}(\rho \otimes P_0^{(1)}) \right] \\ &= \text{tr} \left[\left(\mathbb{I}^{(m+n)} - P_1^{(m)} \otimes P_1^{(n)} \right) \rho \left(\mathbb{I}^{(m+n)} - P_1^{(m)} \otimes P_1^{(n)} \right) \otimes P_1^{(1)} \sqrt{\text{id}}^{(1)} P_0^{(1)} \sqrt{\text{id}}^{(1)} \right. \\ &\quad \left. + \left(P_1^{(m)} \otimes P_1^{(n)} \right) \rho \left(P_1^{(m)} \otimes P_1^{(n)} \right) \otimes P_1^{(1)} \sqrt{\text{id}}^{(1)} \text{NOT}^{(1)} P_0^{(1)} \text{NOT}^{(1)} \sqrt{\text{id}}^{(1)} \right] \end{aligned}$$

$$\begin{aligned}
 &= \text{tr} \left[\left(\mathbb{I}^{(m+n)} - P_1^{(m)} \otimes P_1^{(n)} \right) \rho \left(\mathbb{I}^{(m+n)} - P_1^{(m)} \otimes P_1^{(n)} \right) \right] \text{tr} \left(P_1^{(1)} \sqrt{\mathbb{I}}^{(1)} P_0^{(1)} \sqrt{\mathbb{I}}^{(1)} \right) \\
 &\quad + \text{tr} \left[\left(P_1^{(m)} \otimes P_1^{(n)} \right) \rho \left(P_1^{(m)} \otimes P_1^{(n)} \right) \right] \text{tr} \left(P_1^{(1)} \sqrt{\mathbb{I}}^{(1)} P_1^{(1)} \sqrt{\mathbb{I}}^{(1)} \right) \\
 &= \text{tr} \left[\left(\mathbb{I}^{(m+n)} - P_1^{(m)} \otimes P_1^{(n)} \right) \rho \left(\mathbb{I}^{(m+n)} - P_1^{(m)} \otimes P_1^{(n)} \right) \right] \frac{1}{2} \\
 &\quad + \text{tr} \left[\left(P_1^{(m)} \otimes P_1^{(n)} \right) \rho \left(P_1^{(m)} \otimes P_1^{(n)} \right) \right] \frac{1}{2} \\
 &= \text{tr}(\rho) \frac{1}{2} = \frac{1}{2} = \text{p}(\sqrt{\text{id}} \mathbf{f}).
 \end{aligned}$$

$$(5) \quad \sqrt{\neg} \sqrt{\neg} \alpha \equiv \neg \alpha.$$

By Theorem 4.1(1,3) and because $\sqrt{\text{NOT}}^{(n)} \sqrt{\text{NOT}}^{(n)} \rho = \text{NOT}^{(n)} \rho$, for any $\rho \in \mathfrak{D}(\mathcal{H}^{(n)})$.

$$(6) \quad \sqrt{\neg} \mathbf{f} \equiv \sqrt{\neg} \mathbf{t}.$$

Similar to (2).

$$(7) \quad \neg \sqrt{\neg} \mathbf{f} \equiv \sqrt{\neg} \mathbf{f}; \quad \neg \sqrt{\neg} \mathbf{t} \equiv \sqrt{\neg} \mathbf{t}.$$

By definition of model, by Theorem 4.1(1,3) and because

$$\text{p}(\mathfrak{D} \text{NOT}^{(1)} \mathfrak{D} \sqrt{\text{NOT}}^{(1)} P_0^{(1)}) = \frac{1}{2} = \text{p}(\mathfrak{D} \sqrt{\text{NOT}}^{(1)} P_0^{(1)}) = \text{p}(\mathfrak{D} \text{NOT}^{(1)} \mathfrak{D} \sqrt{\text{NOT}}^{(1)} P_1^{(1)}) = \text{p}(\mathfrak{D} \sqrt{\text{NOT}}^{(1)} P_1^{(1)}).$$

$$(8) \quad \neg \sqrt{\neg} \alpha \equiv \sqrt{\neg} \neg \alpha.$$

By Theorem 4.1(1,3) and because $\mathfrak{D} \sqrt{\text{NOT}}^{(n)} \mathfrak{D} \sqrt{\text{NOT}}^{(n)} \rho = \mathfrak{D} \text{NOT}^{(n)} \rho$, for any $\rho \in \mathfrak{D}(\mathcal{H}^{(n)})$.

$$(9) \quad \sqrt{\neg}(\alpha \wedge \beta) \equiv \sqrt{\text{id}} \mathbf{f}.$$

Similar to (4).

$$(10) \quad \sqrt{\text{id}} \sqrt{\neg} \alpha \equiv \sqrt{\text{id}} \alpha.$$

By Theorem 4.1 (2,3) and because $\text{p}(\mathfrak{D} \sqrt{\mathbb{I}}^{(n)} \mathfrak{D} \sqrt{\text{NOT}}^{(n)} \rho) = \text{p}(\mathfrak{D} \sqrt{\mathbb{I}}^{(n)} \rho)$ for any $\rho \in \mathfrak{D}(\mathcal{H}^{(n)})$.

$$(11) \quad \sqrt{\neg} \sqrt{\text{id}} \alpha \equiv \neg \sqrt{\neg} \alpha.$$

Similar to (10).

$$(12) \quad \sqrt{\text{id}} \sqrt{\neg}(\alpha \wedge \beta) \equiv \sqrt{\neg} \mathbf{f}$$

Similar to (4).

$$(13) \quad \sqrt{\neg} \sqrt{\text{id}}(\alpha \wedge \beta) \equiv \sqrt{\neg} \mathbf{f}$$

Similar to (4). □

Theorem 5.2 shows how the ‘‘Boolean’’ fragment of **HQCL** (formalized in the language \mathcal{L}^C) is a quite weak logic with strongly non-classical features. As happens in the case of most fuzzy logics, conjunctions and disjunctions of **HQCL** are generally non-idempotent. This is, of course, expected in all situations that involve information-transmission, where ‘‘repetita iuvant’’. The failure of commutativity, associativity and distributivity (for \wedge and \vee) seems to be confirmed and justified by a number of examples that concern informal arguments (expressed in natural languages) or semantic situations arising in the languages of art (for instance, in literature or in music). At the same time, it is not easy to find appropriate linguistic models for the genuine quantum computational connectives ($\sqrt{\text{id}}$ and $\sqrt{\neg}$) outside the domain of quantum-information phenomena. Interestingly enough, a suggestion, in this direction, comes from a formal semantics of music.⁵ Consider the case of *musical modulations*, whose characteristic role is creating tonality-changes in a given composition. The starting point may be a *precise* tonality (say, *C* major), which is followed by a situation of *tonal ambiguity*, where different tonalities co-exist in a form that seems to behave like a quantum superposition. Finally one arrives at another tonality, which may be either closely related or distant with respect to the original tonality. From an abstract point of view, such a transformation seems to be similar to what happens when the gate square

⁵See [8].

root of negation is applied twice. A first application of $\sqrt{\text{NOT}}^{(1)}$ to a classical certainty, represented for instance by the bit $|1\rangle$, gives rise to a maximally uncertain *quantum perhaps*: the superposition $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, for which the two canonical truth-values are equally probable. But, then, a second application of the same gate transforms this maximal uncertainty into a different classical certainty, represented by the bit $|0\rangle$.

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