Amit Hagar[*], Giuseppe Sergioli[**]

COUNTING STEPS: A FINITIST APPROACH TO OBJECTIVE PROBABILITY IN PHYSICS

Abstract. We propose a new interpretation of objective probability in statistical physics based on physical computational complexity. This notion applies to a single physical system (be it an experimental set-up in the lab, or a subsystem of the universe), and quantifies (1) the difficulty to realize a physical state given another, (2) the 'distance' (in terms of physical resources) between a physical state and another, and (3) the size of the set of time-complexity functions that are compatible with the physical resources required to reach a physical state from another. This view (a) exorcises 'ignorance' from statistical physics, and (b) underlies a new interpretation to non-relativistic quantum mechanics.
Key-words: probability, ignorance, objectivity, subjectivity, statistical mechanics, quantum mechanics, complexity.

Riassunto: Contare i passi: un approccio finitista alla probabilità oggettiva in fisica. Viene proposta un'interpretazione originale della probabilità oggettiva in fisica basata sul concetto di complessità computazionale. Tale nozione è applicabile a un singolo sistema fisico (sia che si tratti di un sistema fisico relativo ad un esperimento di laboratorio, sia che ci si riferisca, in maniera più generica, ad un arbitrario sottosistema dell'universo), e quantifica (1) la difficoltà di realizzare uno stato fisico in relazione ad un altro, (2) la "distanza" (in termini di risorse computazionali) tra uno stato fisico e un altro, e (3) la dimensione dell'insieme delle funzioni di complessità computazionale che sono compatibili con le risorse fisiche necessarie per raggiungere uno stato fisico a partire da un altro. Questo punto di vista (a) esorcizza l'"ignoranza" dalla fisica statistica, e (b) si basa su una nuova interpretazione della meccanica quantistica non relativistica.
Parole-chiave: probabilità, ignoranza, oggettività, soggettività, meccanica statistica, meccanica quantistica, complessità.

[*] Indiana University, Department of History and Philosophy of Science, Bloomington.
[**] University of Cagliari. The Author has been supported by the Italian Ministry of Scientific Research within the FIRB project "Structures and dynamics of knowledge and cognition", Cagliari unit: F21J12000140001.

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1. Introduction

Probabilistic statements in a deterministic dynamical setting are commonly understood as epistemic (Lewis 1986). Since in such a setting a complete specification of the state of the system at one time – together with the dynamics – uniquely determine the state at later times, the inability to predict an outcome exactly (with probability 1) is predicated on the notion of ignorance, or incomplete knowledge. Such a subjective interpretation is natural in the context of classical statistical mechanics (SM), where a physical state is represented as a point on phase space, and the dynamics is a trajectory in that space, or in the context of Bohmian mechanics, where the phase space is replaced with a configuration space and the ontology is augmented with the quantum potential, but recently it has been suggested as a viable option also in the context of orthodox non-relativistic quantum mechanics (QM), where the state is represented as a vector in the Hilbert space, and the dynamics is a unitary transformation, i.e., a rotation, in that space (Caves, Fuchs, Schack 2002). In all three cases the dynamics is strictly deterministic, and the only difference – apart from the representation of the state – is in the character of the probabilities: in classical SM or Bohmian mechanics they are subsets of phase space (or configuration space) obeying a Boolean structure; in QM they are angles between subspaces in the Hilbert space obeying a non-Boolean structure, whence the famous non-locality, contextuality, and the violation of Bell's inequalities.

Such an epistemic notion of probability in statistical physics appears to many inappropriate. The problem is *not* how lack of knowledge can *bring about* physical phenomena (Albert 2000, p. 64); it can't. Neither is it a problem about ontological vagueness (Hagar 2003). Rather, the problem is that an epistemic interpretation of probability in statistical physics, be it classical SM or QM, turns these theories into a type of statistical inference: while applied to physical systems, these theories become theories about epistemic judgments in the light of incomplete knowledge, and the probabilities therein do not represent or influence the physical situation, but only represent our state of mind (Frigg 2007; Uffink 2011).

The paper proposes a new interpretation of objective physical probability as an alternative to this epistemic notion, and is structured as follows. In section 2 we spell out the basic assumptions behind our proposal. In section 3 we present our new model of objective probability that quantifies how hard it is to realize a physical state, and measures (in terms of physical resources) the 'distance' between any such pair of states. In section 4 we apply our new interpretation to classical SM, and explain why we believe it is also applicable to QM.

In doing so we introduce a new interpretation to the probabilities of QM. Section 5 explains how our finitist approach can distinguish between classical and quantum probabilities according to their different complexity-induced measures. Section 6 concludes.

2. Assumptions

We start by spelling out the five basic assumptions that underlie our models. They are *Determinism*, $P \subset EXPTIME$, *Finitism*, *Discreteness*, and *Locality*. These assumptions are working hypotheses in the framework from which our interpretation of probability stems, namely, physical computational complexity. In this framework (Geroch, Hartle 1986; Pitowsky 1990; 1996), the performance of physical systems is analyzed with notions and concepts that originate in computational complexity theory, by approximating dynamical evolutions with a discrete set of computational steps to an arbitrary degree of accuracy. These assumptions help us delineate the two probability spaces in our models: the space of physically allowable states, and the space of physically allowable dynamical evolutions.

2.1. Determinism

Our model rests on the assumption of strict determinism. This assumption follows from the strong physical Church-Turing thesis (PCTT henceforth)[1], according to which actual dynamical evolutions of physical systems in our world can be regarded as computations carried by deterministic Turing machines. Agreed, some physical theories do allow *in principle* for non-Turing-computable phase trajectories (trajectories that cannot be represented by recursive functions), and, in addition, there exists a vast literature on the physical possibility of supertasks and 'hypercomputation', that aims to show that Turing-computability is not a natural property, and need not apply *a priori* in the physical world. Nevertheless, if the strong PCTT holds, then *as a contingent matter of fact*, non-Turing-computable trajectories are ruled out, and the above, rather contrived, counterexamples are not realizable in the actual

[1] The *physical* Church-Turing thesis is logically independent of the original Church-Turing thesis. See e.g., Shagrir, Pitowsky (2003).

universe[2]. In what follows we thus disregard naked singularities, closed time-like curves, non-globally hyperbolic spacetime models, ill-posed problems, divergences, and the like, adhering to the idea that every dynamical evolution takes a physical state to one and only one physical state[3].

2.2. P $\subset$ ExpTime

The fact that each computation requires physical resources (energy and time) that increase with the size (the number of degrees of freedom) of the system allows us to classify different dynamical evolutions as either 'easy' (i.e., having polynomial time-complexity such as $O(n^c)$) or 'hard' (i.e., having exponential time-complexity such as $O(c^n)$)[4].That there exists a meaningful difference between different degrees of time-complexity within each class is the consequence of the Time Hierarchy Theorems (Hartmanis, Stearns 1965). While these theorems provide no means to relate deterministic and non-deterministic complexity, or time and space complexity, they still allow us to assume that given more time, a Turing machine can solve more problems. For example, there are problems that can be solved with n^2 time but not n time. Here we make an even stronger assumption, which is a working hypothesis in computer science, that there exists also a meaningful difference between the two classes themselves.

2.3. Finitism

Assumption (2.1) allows us to apply the machinery of complexity theory to dynamical evolutions, by treating them as computations. Assumption (2.2) allows us to classify states (and the dynamical evolutions that realize them) as 'easy' or 'hard'. Assumption (2.3) allows us to impose upper and lower bounds on the set of all possible dynamical evolutions in the actual universe, based on the assumption that the total energy in the universe is finite.

[2] So far there are two such counterexamples: Pour-el and Richards's (1989) wave equation in 3 dimensions and Pitowsky's (1990) spacetime model that allows finite-time execution of an infinite number of computational steps. See also Hogarth (1994) for an elaboration on the latter, and Earman, Norton (1993) for further discussion.

[3] Note that from a strictly dynamical perspective, quantum *dynamics* is fully deterministic: Schrödinger's equation takes any quantum state to one and only one quantum state.

[4] Here n is the input size – in our case the dimension of the system at hand, and c is a (bounded rational, as we shall assume below) coefficient.

2.4. Discreteness

Assumption (2.4) allows us to discretize the set of the physically allowable dynamical evolutions. Two facts warrant the elimination of real coefficients in our classification of dynamical evolutions into time-complexity classes. First, each dynamical evolution is governed by a Hamiltonian (the total energy function). Second, if energy is finite and bounded from above (assumption 2.3), it is impossible to resolve arbitrary energy differences between any two Hamiltonians (Hilgevoord 1998, p. 399). Note that here we make the finitist claim that limitations on resolution reflect actual discreteness in nature, and are not simply a matter of practical constraints.

2.5. Locality

Finally, and consistent with the current state of affairs in physics, in physically realizing the Hamiltonians that govern the dynamical evolutions, we allow only local interactions.

3. A Possible Probability Model

The above assumptions allow us to propose a possible model for objective physical probability. We do not claim that this model is unique, optimal, or in any sense canonical. Our purpose is only to demonstrate that it is possible to define a finite notion of objective probability in physics on the basis of considerations from physical computational complexity.

The model is constructed on the space of all possible dynamical evolutions that connect any two states of a given physical system with a given number of degrees of freedom n in a given moment in time t, confined to a given energy shell E. This triplet, i.e., the number of degrees of freedom n, time t, and energy E is required for the precise definition of probability. Given such a triplet, and our assumptions (2.1-2.5), we construct a probability space out of a functional that relates the power ($P = E/t$) of a computation – seen as a dynamical evolution from one state to another – with the relative size of the set of the possible dynamical evolutions that are compatible with it. Our probability function is thus a distance measure on the above functional, that quantifies how hard it is to realize a state, or how far a given system is from that state, in terms of the physical resources available to it, relative to the required resources.

With this model we suggest to interpret objective probability as a physical magnitude that quantifies how hard it is to realize a physical state, given a triplet of physical resources (energy, time, space). Equivalently, this magnitude quantifies how 'far' a given physical system is from a certain state in terms of the physical resources available to it, relative to those required for that state's realization. The following model captures this interpretation, while formally admitting the constraints of probability theory.

- Take any physical system with dimension n in a given energy state E and in a given moment in time t, and let Ω be the bounded and discrete set of possible dynamical evolutions obeying the current laws of physics, whose time-complexity is either polynomial or exponential ('easy' or 'hard'), that may govern the system's behavior. The set Ω contains all possible *dynamical evolutions* that can realize a single actual state.
- Given a certain couple (n, $P = \frac{E}{t}$), where n is the dimension of the state, E is the total possible energy, and t is the total possible time (hence P is the power allowed for the computation), we consider the set

$$S_{\bar{n}} = \{g_n \in \Omega / O(g_{\bar{n}}) \leq N(\mathrm{P})\} \tag{1}$$

where g_n is a dynamical evolution that for a given n 'consumes' at most the resources E in time t with the number of computational steps N[5].

- F is the σ-algebra of S, i.e., a non-empty class of subsets of S, containing S itself, the empty set, and closed under the formation of complements, finite unions, and finite intersections. The elements of F are dynamical evolutions with a combined time-complexity, either exponential or polynomial. F is thus a subset of the power set of S, and is bounded and discrete.

[5] By 'consumes' we mean the following. Take an arbitrary computation. Each computational step 'costs' the same amount of time; but if, as in our case, the *total* time allowed for the computation is fixed, the difference in time-complexity is cashed out in terms of the difference in the frequency of the computation, i.e., the time-difference between any two computational steps. Thus, for a given n and for a given t, the higher the degree of time-complexity of the function, the higher the frequency of the computation. Since higher frequency means higher power, by setting a bound on P, one immediately sets a bound on N, the number of computational steps allowed for the computation, and subsequently, a bound on $|S_{\bar{n}}|$ the number of time-complexity functions that can realize the computation.

Our probability measure p is given by the mapping:

$$\forall A \in F: p_{\{n_A, \mathrm{P}_A\}}(A) = |A|/|S| \tag{2}$$

Where P_A is the available power. To calculate this magnitude we embed it in a continuous function of the general convex form $\mathrm{P} = \left(\frac{1}{n^\alpha}N\right)^\beta$, where α and β and are free parameters.

By construction $p(A) \in [0,1], p(\emptyset) = 0, p(S) = 1$ and p is additive: $\forall A, B \in F$ such that $A \cap B = \emptyset, p(A \cup B) = p(A) + p(B)$. It can be shown that under a suitable choice of the parameters α and β, the area below the curve $\mathrm{P} = \left(\frac{1}{n^\alpha}N\right)^\beta$ can be seen as a probability space, satisfying further constraints imposed by the axioms of probability theory (e.g., independence and conditional probability).

4. Ignorance of What?

Consistent with our goal to turn epistemic probability in statistical physics into an objective one, the notion of physical probability here proposed has nothing to do with one's credence or degrees of belief. It measures, as we have seen, the difficulty (in terms of physical resources) to realize the transition from one state to another. The more probable a state, the easier it is to reach it from a given state with a given amount of resources.

To see how this notion of probability can turn subjective 'ignorance' in statistical physics into an objective feature of the world, we propose the following intuition.

4.1. Classical Mechanics

Our probability notion is intimately related to the notion of measurement resolution. Take classical statistical mechanics, where one introduces a distinction between micro-states and macro-states. The evolution of the former on phase space is constrained by Liouville's theorem, that tells us that a region of phase space (call it 'a blob'), occupied by a set of micro-states all compatible with a certain macro-state, may change is shape but not its volume. The 'evolution' of the latter is dictated by the kind of measurements we make, i.e., by the different partitions we impose on phase space. These two evolutions are *independent*[6], and they allow us to define the transition probability of a physical system from one macro-state to another as the partial overlap between the blobs and the macro-states:

$$p\big([M_1]_{t_1} | [M_0]_{t_0}\big) = \mu\big(B_{t_1} \cap [M_1]\big) \quad (3)$$

This means that the probability that a system that starts at a macro-state $[M_0]$ at time t_0 (when the size of the dynamical blob B completely saturates the volume $[M_0]$) will end in a macro-state $[M_1]$at time t_1, is given by the partial overlap (the relative size) μ of the dynamical blob B at t_1 with the macro-state $[M_1]$. Note that there is nothing subjective in this kind of transition probability. 'Ignorance' here simply means lack of resolution power, i.e., lack of precision or lack of control, which is expressed by the relation between dynamical blobs and macro-states, both of which are objective features of the physical world[7].

One can describe the evolution of a dynamical system either by following its dynamical blob, or, equivalently by following the macro-states to which the exact state belongs. In the first description probability signifies lack of precision; in the second, lack of control. We have already shown that our probability measure describes the amount of missing resources for an exact description in the first case. In the second case, we can define our probability as an objective physical magnitude, a transition probability between two macro-states M_0 and M_1, that signifies how 'far' is M_1 from M_0 where the 'distance' $p(M_0,M_1)$ is defined in terms of the physical resources (energy, space, and time) that an observer who observes M_0 *has*, relative to what she *needs* in order to observe M_1.

In this sense, probability is an objective measure of the *difficulty* to produce the macro-state M_1 from the macro-state M_0 given the physical resources (energy, space, and time) at one's disposal. Moreover, this measure is *identical*, conceptually and formally, to the one used in the foundations of

[6] See Hemmo, Shenker (2012) for the trouble one gets into when one ignores this independence.

[7] An anonymous referee raises doubts whether the partition of phase space into macro-states is an objective feature of the world. In reply, the point here is that the carving of the phase space into macro-states is not subjective because it is an objective physical fact about observers like us that they are correlated with the world in such a way that leads to such a carving. We assume here that the theoretician chooses certain macro-states as meaningful (i.e., measurable) only because she is such an observer for which these macro-states are meaningful (i.e., measurable). Of course, a complete account of this choice may require a physicalist solution to the mind-body problem which we do not attempt to solve here. But note that the *objectivity* of this choice is independent of what such a solution would look like.

statistical mechanics, as one can interpret any probability less than *1* as signifying the lack of physical resources that can allow one to partition phase space into a macro-state more accurately in such a way that it will include *all* of the dynamical blob.

The empirical conjecture we make, over and above the requirement for conformity with the observed relative frequencies, is that this relative volume of the dynamical blob in M_1 (i.e., p in (3)) should be a function of the physical resources we have relative to what we need in order to observe M_1 with certainty. This conjecture is *in principle* testable in many scenarios within control theory, where one is trying to steer a physical process to a desirable outcome. A similar situation holds, we believe, in the case of QM[8], which is a little more complicated conceptually.

4.2. Quantum Mechanics

QM uses the mathematical formalism of the Hilbert space (Beltrametti *et al*. 2010; Dalla Chiara *et al*. 2009), which is a vector space over the complex numbers equipped with an inner product. How can a finite and discrete model such as ours be compatible with such a continuous structure? In particular, how can a finite and discrete probability space such as ours even approximate the notion of 'inner product', which requires transitive ordering?

Here we do not offer a complete answer, but only allude to existing consistency proofs that show that such an approximation of the continuous by the discrete is not so far fetched as one would have thought. The key point is that our model need not reproduce *all* the continuous structure; it only needs to reproduce those parts of the continuum structure that have *observable* consequences.

Such proofs rely on finite fields, also called Galois fields, which are of the type $GF(p^n)$ (where $GF(p^n)$ is the field of integers $\mathbb{Z}$ *mod* (p^n), p is a prime, and $n \in \mathbb{Z}$) (Van Bendegem 2010). The key challenge in these proofs is to find such a finite field, a sufficiently large portion of which is 'like the real number system' with which one could describe the observable universe, i.e., a range between 10^{-13} and 10^{27} (the range between one Fermi and the distance to the farthest known object in the universe). Clearly there is no difficulty in finding enough points from a field $GF(p^n)$ provided p is large enough. In

[8] Albert (2000, Ch. 5) offers a similar conjecture when he proposes that the probabilities of SM supervene on transition probabilities of a more fundamental collapse dynamics. Our view is deterministic, hence excludes collapse, but we too suggest that probabilities in statistical physics are dynamical transition probabilities. In our story, however, they supervene on time-complexity and relative physical resources.

particular (Coire 1959), to fill the range of the observable universe a prime $p \approx 10^{10^{81}}$ is sufficient.

But to approximate the real number system, such a subset of the finite field, no matter how huge, must also be transitively ordered, a very non-trivial constraint given the periodicity of the finite field. And yet, what has been shown is that if the prime is chosen to have the form[9]:

$$p = \left(8x \prod_{i=1}^{k} q_i \ - 1 \right)$$

where x is an odd integer and $\prod_{i=1}^{k} q_i$ is the product of the first k-odd primes, then -1_q is "negative" and *2* and the first k-odd primes are 'positive'. For such a prime the first N integers for large N can be (locally) transitively ordered and consequently the geometry in that neighborhood would appear to be like ordinary Euclidean plane, up to very large (and down to very small) distances (Morris 1974).

Later work (Reisler, Smith 1969) incorporated these abstract considerations from number theory into physics by developing concepts such as order, norm, metric, and inner product over the above subset of the total finite field in which transitive order could be defined. With these 'extensions' it became clear that a finite discrete space behaves locally (albeit not globally) like the standard conventional continuum. This insight, namely that a discrete description of physical phenomena in the neighborhood of the 'ordered' subset of the total field is locally indistinguishable from the standard continuum description, was also repeated by J. Schwinger (2001, p. 84), and has recently reappeared in attempts to approximate the continuum of the Hilbert space with a vector space constructed over a specific Galois field $GF(p^2)$ of the sort described above (Hanson *et al*. 2013a). Insofar as this vector space can approximate (locally) the notion of an inner product, and can support showcase quantum algorithms (Hanson *et al.* 2013b), these attempts have also succeeded in reproducing the empirical content of non relativistic quantum mechanics from an underlying finite and discrete structure.

Once the conceptual difficulty of approximating (at least locally) the continuous with the discrete is removed, our proposal can serve as a basis for a

[9] The existence of a prime of this form is guaranteed by Dirichlet's theorem, that states that for any two positive co-prime integers *a* and *d*, there are infinitely many primes of the form $a+nd$, where *n* is a non negative integer. In other words, there are infinitely many primes which are congruent to *a mod d*.

new interpretation of QM. Recall that on the subjective view of QM, the quantum state is treated as a state of knowledge, and quantum probabilities (calculated by the Born rule) are interpreted as “gambling bets” of agents on results of experiments, *à la* Ramsey-De Finetti (Fuchs 2010). In contrast, in Bohmian mechanics, the alternative epistemic approach in the foundations of QM, the probabilities are for particles to have certain positions; they signify our ignorance thereof.

Our new idea about probability simply avoids this debate altogether by supplying a possible third way: what quantum probabilities are probabilities *for* is neither the positions of particles, nor the gambling bets of learned observers. Rather, quantum probabilities simply quantify how hard it is to realize a physical state; they measure the ‘distance’ between the current state of a physical system and any other state thereof, given the resources (energy/time) that are available to that system at that moment. This alternative allows us to interpret quantum probabilities as objective deterministic chances (and in so doing to turn QM once again into a physical theory about the world), without having to support nonlocal hidden variables.

5. Classical vs. Quantum Probabilities

In the approach presented here, the origins of both quantum and classical probabilities is identical – they both stem from objective deterministic chances which supervene on time-complexity classes and relative availability of physical resources. The crucial point is that despite the lack of *metaphysical* difference, we can still distinguish between quantum and classical probabilities in structural, or formal, terms. Instead of “hidden variables” vs. “quantum indefiniteness” we suggest a quantitative difference in *measure*, which in our case is complexity-induced. Indeed, it is a working hypothesis within quantum information scientists that any classical computation that would be harnessed for the simulation of quantum phenomena would do so inefficiently[10].

Our approach can easily accommodate such a putative difference: the notion of probability we propose here is *defined* as the relative size of the set of time-complexity classes that can realize a physical state. That quantum and classical probabilities share the same origins need not entail that for *every* physical state the above relative size is also identical. Quantum dynamical evolutions may “consume” (in the sense developed in fn. (5)) less

[10] This conjecture was first voiced by Richard Feynman (1982). Computer scientists have formalized it as BPP $\subseteq$ BQP (Aaronson 2009).

resources than classical ones, and so the probability of some physical states may as well be different when realized by quantum or by classical dynamics. We suggest to view the violations of Bell's inequality as designating exactly this difference; a difference in complexity, not in metaphysics (Buhrman *et al*. 1998, Beltrametti *et al*. 2012b). Note, moreover, that such a criterion is completely in accord with our current empirical knowledge, and yet, contrary to its metaphysical counterpart, it leaves open the question of the universality of quantum theory[11].

6. Conclusion: Probability as Distance Measure

The 'distance' between any two physical states (in terms of the relative physical resources required for such a transition) satisfies Kolmogorov's axioms. At least mathematically, therefore, our model is worthy of the name 'probability'. It also explains away ignorance by tying error (in the preparation of the initial state) to probability (of the desired state), and by supervening this probability on time-complexity and physical resources. In the classical context, it appears to be a natural physical interpretation of the epistemic probabilities that arise in statistical mechanics. Here we put forward the (empirical!) conjecture that such a distance measure reproduces the quantum Born rule hence can be regarded a novel interpretation of quantum probabilities.

We emphasize again that we are only proposing a new *interpretation* to the meaning of probability: instead of interpreting probability as an epistemic measure of ignorance (which is the standard way in a deterministic dynamical context), we propose an interpretation in terms of the distance (in terms of the relative physical resources) between an actual state and an ideal one. In this sense our proposal is only qualitative. Moreover, we do not pretend in any way to go beyond objective probabilities in statistical physics in our interpretation. Whatever problems exist in connecting these with the *ordinary* notion of probability, namely the connection to relative frequencies, or to betting behavior, also exist in our interpretation, and we do not purport to solve them here.

Concluding, we have argued that he amount of physical resources that separate two physical states is an objective feature of the world, and that computational complexity theory allows us to map this feature onto [0,1]. This mapping, we claim, has all the characteristics of a discrete probability

[11] Our probability measure depends on the dimension of the system, which appears to be a key factor in the open problem of scaling-up quantum information processing devices.

function, and can be interpreted as a measure of precision and control that one has in one's disposal in the resolution of the physical state during the process of measurement. This measure can furthermore ground a new interpretation of probability in statistical physics, both classical and quantum, as an objective, transition probability between any two physical states.

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