Representing fuzzy structures in quantum computation with mixed states

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***Abstract*—In this work we introduce a particular kind of quantum operations called *polynomial quantum operations* that allow us to represent the basic operations of the standard Product MV-algebra. Consequently, these operations can be treated as quantum computational gates in the powerful model of quantum computation given by "quantum operations - density operators".**

Keywords: Quantum computation, quantum operations, fuzzy logic.

I. Introduction

In the usual representation of quantum computational processes, a quantum circuit is identified with an appropriate composition of *quantum gates*, i.e. unitary operators acting on pure states of a convenient (n-fold tensor product) Hilbert space. Consequently, quantum gates represent time reversible evolutions of pure states of the system. Nonetheless, this constrain is unduly restrictive. Apparently, it does not encompass states whose information is non-maximal, such as states whose preparation is unknown. Moreover, there are relevant physical processes that cannot be represented by unitary evolutions, such as measurements in the middle of a process, decoherence and so on. Several authors [1], [3], [5], [8], [12] considered a more general model of quantum computational processes called *quantum computation with mixed states*, where pure states and unitary operators are replaced by density operators and quantum operations, respectively. In this case, time evolution is, in general, no longer reversible. Further, such a model will allow us to represent, in a probabilistic way, the operations of the standard PMV-algebra [6], [10] as quantum operations. This paper is organized as follows: in Section II we provide some basics; in Section III we introduce the *polynomial quantum operations* and we prove a representation theorem which allows us associate a polynomial quantum operation to every element of a particular class of polynomials; finally, in Section IV, we show how to represent the Product t-norm, the Łukasiewicz negation and the Łukasiewicz sum as quantum operations.

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II. Basic notions

Let H be an arbitrary complex Hilbert space. We denote by $\mathcal{L}(H)$ the vector space of all linear operators on H and by $\mathcal{D}(H)$ the set of all density operators. Standard quantum computing is based on quantum systems with finite dimensional Hilbert spaces, specially $\mathbb{C}^2$, the two-dimensional state space of a *quantum bit*. A quantum bit or *qbit*, the fundamental concept of quantum computation, is a pure state in the Hilbert space $\mathbb{C}^2$. The standard orthonormal basis $\{|0\rangle, |1\rangle\}$ of $\mathbb{C}^2$ where $|0\rangle = (1,0)$ and $|1\rangle = (0,1)$ is called the *logical basis*. Thus, pure states $|\psi\rangle$ in $\mathbb{C}^2$ are coherent superpositions of the the basis vectors: $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$, with complex coefficients such that $|c_0|^2 + |c_1|^2 = 1$.

Generalizing for a positive integer n, n-*qbits* are pure states represented by unit vectors in $\otimes^n\mathbb{C}^2$. A special basis, called the 2^n-*computational basis*, is chosen for $\mathbb{C}^{2^n}$. More precisely, it consists of the 2^n orthogonal states $|\iota\rangle$, $0 \leq \iota \leq 2^n$ where ι is in binary representation and $|\iota\rangle$ can be seen as the tensor product of states $|\iota\rangle = |\iota_1\rangle \otimes |\iota_2\rangle \otimes \ldots \otimes |\iota_n\rangle$ where $\iota_j \in \{0,1\}$. A pure state $|\psi\rangle \in \otimes^n\mathbb{C}^2$ is a superposition of the basis vectors $|\psi\rangle = \sum_{\iota=1}^{2^n} c_\iota |\iota\rangle$ with $\sum_{\iota=1}^{2^n} |c_\iota|^2 = 1$.

In general, a quantum system is not in a pure state. For, a physical system is always interacting with the environment, and therefore phenomena such as decoherence and noise come into play. These physical systems whose information is non-maximal are named *mixed states*, and they are described by *density operators*. A density operator is represented on the 2^n-dimensional complex Hilbert space by an Hermitian (i.e $\rho^\dagger = \rho$) positive operator with unit trace, $tr(\rho) = 1$. Due to the fact that Pauli matrices:

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$$\sigma_0 = I;\ \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};\ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix};$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};$$

where $I = I^{(2)}$ is the 2×2 identity matrix, form a basis for the set of operators on $\mathbb{C}^2$, an arbitrary density operator ρ for n-qbits may be represented in terms of tensor products of them in the following way:

$$\rho = \frac{1}{2^n} \sum_{\mu_1 \ldots \mu_n} P_{\mu_1 \ldots \mu_n} (\sigma_{\mu_1} \otimes \ldots \otimes \sigma_{\mu_n})$$

where $\mu_i \in \{0, x, y, z\}$ for each $i = 1 \ldots n$. As usual, we have chosen units such that $\hbar = 1$. The real expansion coefficients $P_{\mu_1 \ldots \mu_n}$ are given by $P_{\mu_1 \ldots \mu_n} = tr(\sigma_{\mu_1} \otimes \ldots \otimes \sigma_{\mu_n} \rho)$. Since the eigenvalues of the Pauli matrices are ± 1, the expansion coefficients satisfy $|P_{\mu_1 \ldots \mu_n}| \leq 1$.

Taking into account the Born roule, for $\rho \in \mathcal{D}(\otimes^n \mathbb{C}^2)$ we define the probabilty value of ρ as $\mathtt{p}(\rho) = tr(P_1^{(n)} \rho)$, i.e. the expectation value of ρ in the state $P_1^{(n)}$, where $P_1^{(n)} = (\otimes^{n-1} I) \otimes |1\rangle\langle 1|$. Let $\rho \in \mathcal{D}(\mathbb{C}^2)$ such that $\rho = \frac{1}{2}(I + r_x \sigma_x + r_y \sigma_y + r_z \sigma_z)$, i.e.

$$\rho = \frac{1}{2} \begin{pmatrix} 1 + r_z & r_x - i r_y \\ r_x + i r_y & 1 - r_z \end{pmatrix} = \begin{pmatrix} 1 - \alpha & \beta \\ \beta^* & \alpha \end{pmatrix}.$$

Interestingly enough, any real number λ ($0 \leq \lambda \leq 1$) uniquely determines a density operator ρ_λ of the following form:

$$\rho_\lambda = (1-\lambda) P_0 + \lambda P_1 = \frac{1}{2}(I + (1-2\lambda)\sigma_z) = \begin{pmatrix} 1 - \lambda & 0 \\ 0 & \lambda \end{pmatrix}.$$

It is easy to see that, if $\rho \in \mathcal{D}(\mathbb{C}^2)$, then $\mathtt{p}(\rho) = \frac{1 - r_z}{2}$ and $\mathtt{p}(\rho_\lambda) = \lambda$. Thus each density operator ρ in $\mathcal{D}(\mathbb{C}^2)$ can be written as

$$\rho = \begin{pmatrix} 1 - \mathtt{p}(\rho) & a \\ a^* & \mathtt{p}(\rho) \end{pmatrix}.$$

In the usual representation of quantum computational processes, a quantum circuit is identified with an appropriate composition of *quantum gates*, mathematically represented by *unitary operators* acting on pure states of a convenient (n-fold tensor product) Hilbert space $\otimes^n \mathbb{C}^2$ [11]. In other words, standard quantum computation is mathematically founded on "*qbits-unitary operators*".

A *quantum operation* [9] is a linear operator $\mathcal{E} : \mathcal{L}(H_1) \to \mathcal{L}(H_2)$ representable as $\mathcal{E}(\rho) = \sum_i A_i \rho A_i^\dagger$ where A_i are operators satisfying $\sum_i A_i^\dagger A_i = I$ (Kraus representation Theorem [9]). If A_i are unitary operators, the corespondent quatum operation is named *unitary quantum operation*. It can be seen that a quantum operation maps density operators into density operators. The new model "*density operators-quantum operations*" also called "*quantum computation with mixed states*" ([1], [12]) is equivalent in computational power to the standard one but gives a place to irreversible processes as measurements in the middle of the computation.

III. Polynomial quantum operations

In this Section, we represent in a probabilistic way some classes of polynomials as quantum operations. First of all, we introduce some notations and preliminary definitions. The term *multi-index* denotes an ordered n-tuple $\alpha = (\alpha_1, \ldots \alpha_n)$ of non negative integers α_i. The *order* of α is given by $|\alpha| = \alpha_1 + \ldots\ldots + \alpha_n$.
If $\mathbf{x} = (x_1, \ldots, x_n)$ is an n-tuple of variables and $\alpha = (\alpha_1, \ldots \alpha_n)$ a multi-index, the monomial $\mathbf{x}^\alpha$ is defined by $\mathbf{x}^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$. In this language a real polynomial of order k is a function $\mathtt{p}(\mathbf{x}) = \sum_{|\alpha| \leq k} a_\alpha \mathbf{x}^\alpha$ such that $a_\alpha \in \mathbb{R}$.
Let $\mathbf{x} = (x_1, \ldots, x_n)$ and k be a natural number. Then, we consider the set $D_k(\mathbf{x})$ defined as

$$D_k(\mathbf{x}) = \{(1-x_1)^{\alpha_1} x_1^{\beta_1} \ldots (1-x_n)^{\alpha_n} x_n^{\beta_n} : \alpha_i + \beta_i = k,\ 1 \leq i \leq n\}.$$

Lemma 3.1: Let $\mathbf{X}_1, \ldots, \mathbf{X}_n$ be a family of matrices such that

$$\mathbf{X}_i = \begin{pmatrix} 1 - x_i & b_i \\ b_i^* & x_i \end{pmatrix}$$

and let us consider a tensor product $\mathbf{X} = (\otimes^k \mathbf{X}_1) \otimes (\otimes^k \mathbf{X}_2) \otimes \ldots \otimes (\otimes^k \mathbf{X}_n)$. Then we have that
$Diag(\mathbf{X}) = D_k(x_1, \ldots, x_n)$.

Proof: By induction on k we can prove that $Diag(\bigotimes^k X_i) = \{h_1 h_2 \ldots h_k : h_j \in \{(1 - x_i), x_i\},$
$1 \leq j \leq k\} = \{(1 - x_1)^\alpha x_1^\beta : \alpha + \beta = k\}$.

Thus, $Diag((\otimes^k \mathbf{X}_1) \otimes (\otimes^k \mathbf{X}_2) \otimes \ldots \otimes (\otimes^k \mathbf{X}_n)) =$
$= \{(1 - x_1)^{\alpha_1} x_1^{\beta_1} \ldots (1 - x_n)^{\alpha_n} x_n^{\beta_n} : \alpha_i + \beta_i = k,$
$i \in \{1, \ldots, n\}\}$. Whence our claim follows.

■

Lemma 3.2: Let $\mathbf{x} = (x_1, \ldots, x_n)$ and k be a natural number. Given any monomial $\mathbf{x}^\alpha$ such that $|\ \alpha\ | \leq k$, we have that:

1) $\mathbf{x}^\alpha = \sum_{\mathbf{y} \in D_k(\mathbf{x})} \delta_\mathbf{y} \mathbf{y}$;
2) $1 - \mathbf{x}^\alpha = \sum_{\mathbf{y} \in D_k(\mathbf{x})} \gamma_y \mathbf{y}$;

where δ_y and γ_y lies in $\{0, 1\}$.

Proof: For each $i \in \{1, \ldots, n\}$, consider the matrix $\mathbf{X_i}$ given by

$$\mathbf{X}_i = \begin{pmatrix} 1 - x_i & 0 \\ 0 & x_i \end{pmatrix}.$$

Let us prove 1. Let $\mathbf{x}^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$ such that $|\ \alpha\ | \leq k$. Thus, there exist $s_1, \ldots, s_n$ such that $\alpha_i + s_i = k$.
Let $\mathbf{W} = (\otimes^{s_1} \mathbf{X}_1) \otimes (\otimes^{s_2} \mathbf{X}_2) \otimes \ldots \otimes (\otimes^{s_n} \mathbf{X}_n)$ and let consider the matrix $\mathbf{W}\mathbf{x}^\alpha$. In view of Lemma 3.1, $Diag(\mathbf{W}\mathbf{x}^\alpha) \subseteq D_k(x_1, \ldots, x_n)$ since every element in

$Diag(\mathbf{W}\mathbf{x}^\alpha)$ is a monomial of order nk. Further, since $tr(\mathbf{W}\mathbf{x}^\alpha) = (tr\mathbf{W})\mathbf{x}^\alpha = 1\mathbf{x}^\alpha = \mathbf{x}^\alpha$, we have that $\mathbf{x}^\alpha = tr(\mathbf{W}\mathbf{x}^\alpha)$ is the required polynomial expansion.

Now we prove 2. Let $\mathbf{X} = (\otimes^k \mathbf{X}_1) \otimes (\otimes^k \mathbf{X}_2) \otimes \ldots \otimes (\otimes^k \mathbf{X}_n)$. By Lemma 3.1 $Diag(\mathbf{X}) = D_k(x_1, \ldots, x_n)$ and $tr(\mathbf{X}) = 1$. Taking into account that $\mathbf{x}^\alpha = \sum_{\mathbf{y}\in D_k(\mathbf{x})} \delta_\mathbf{y}\mathbf{y}$, we define $\gamma_\mathbf{y} = 1$ if $\delta_\mathbf{y} = 0$ and $\gamma_\mathbf{y} = 0$ if $\delta_\mathbf{y} = 1$. Therefore, $1 = tr(\mathbf{X}) = \sum_{\mathbf{y}\in D_k(\mathbf{x})} \delta_\mathbf{y}\mathbf{y} + \sum_{\mathbf{y}\in D_k(\mathbf{x})} \gamma_y\mathbf{y} =$
$= \mathbf{x}^\alpha + \sum_{\mathbf{y}\in D_k(\mathbf{x})} \gamma_y\mathbf{y}$ and $1 - \mathbf{x}^\alpha = \sum_{\mathbf{y}\in D_k(\mathbf{x})} \gamma_y\mathbf{y}$. ■

Definition 3.1: A quantum operation $\mathcal{P} : \mathcal{L}(\otimes^{nk}\mathbb{C}^2) \to \mathcal{L}(\otimes^{nk}\mathbb{C}^2)$ is called *polynomial quantum operation* iff there exists a polynomial $P(x_1, \ldots, x_n)$ such that for each n-tuple $(\sigma_1, \ldots, \sigma_n)$ in $\mathcal{D}(\mathbb{C}^2)$ we have that:

$$\mathtt{p}(\mathcal{P}((\otimes^k\sigma_1) \otimes \ldots \otimes (\otimes^k\sigma_n))) = P(\mathtt{p}(\sigma_1), \ldots, \mathtt{p}(\sigma_n)).$$

Theorem 3.1: Let $\mathbf{x} = (x_1, \ldots, x_n)$ be an n-tuple of variables and consider the set $D_k(\mathbf{x})$. Let $P(\mathbf{x}) = \sum_{\mathbf{y}\in D_k(\mathbf{x})} a_\mathbf{y}\mathbf{y}$ be a polynomial such that $\mathbf{y} \in D_k(\mathbf{x})$, $0 \le a_\mathbf{y} \le 1$ and the restriction $P(\mathbf{x}) \restriction_{[0,1]^n}$ satisfies that $0 \le P(\mathbf{x}) \restriction_{[0,1]^n} \le 1$.
Then there exists a polynomial quantum operation
$\mathcal{P} : \mathcal{L}(\otimes^{nk}\mathbb{C}^2) \to \mathcal{L}(\otimes^{nk}\mathbb{C}^2)$ such that for each n-tuple $\sigma = (\sigma_1, \ldots, \sigma_n)$ in $\mathcal{D}(\mathbb{C}^2)$

$$\mathtt{p}(\mathcal{P}((\otimes^k\sigma_1) \otimes \ldots \otimes (\otimes^k\sigma_n))) = P(\mathtt{p}(\sigma_1), \ldots, \mathtt{p}(\sigma_n)).$$

Moreover, $\mathcal{P}((\otimes^k\sigma_1) \otimes \ldots \otimes (\otimes^k\sigma_n)) = (\frac{1}{2^{nk-1}} \otimes^{nk-1} I) \otimes \rho_{P(\mathtt{p}(\sigma_1),\ldots,\mathtt{p}(\sigma_n))}$.

Proof: Let $\sigma_1, \ldots, \sigma_n$ density operators of $\mathbb{C}^2$. Assume that for any σ_i

$$\sigma_i = \begin{pmatrix} 1 - x_i & b_i \\ b_i^* & x_i \end{pmatrix}.$$

Hence, $\mathtt{p}(\sigma_i) = x_i$. It is clear that $\sigma = (\otimes^k\sigma_1) \otimes \ldots \otimes (\otimes^k\sigma_n)$ is a matrix of order $2^{nk} \times 2^{nk}$ and, by Lemma 3.1,
$Diag(\sigma) = D_k(x_1, \ldots, x_n)$. Thus, each $\mathbf{y} \in D_k(\mathbf{x})$ can be seen as the (i,i)-th entry of $Diag(\sigma)$. Further, the polynomial $P(\mathbf{x}) = \sum_{\mathbf{y}\in D_k(\mathbf{x})} a_\mathbf{y}\mathbf{y} = \sum_{j=1}^{2^{nk}} a_j\mathbf{y}_j$ is such that every $\mathbf{y}_j$ is the (j,j)-th entry of $Diag(\sigma)$. Let, now, $\mathbf{y}_{j_0} \in Diag(\sigma)$.

a) We want to place $a_{j_0}\mathbf{y}_{j_0}$ in the $(2s, 2s)$-th entries of a $2^{nk} \times 2^{nk}$ matrix.
Let us consider the $2^{nk} \times 2^{nk}$ matrix $A_{j_0}^{2s} = \sqrt{\frac{a_{j_0}}{2^{nk-1}}} D_{j_0}^{2s}$ such that $D_{j_0}^{2s}$ has 1 just in the $(2s, j_0)$-th entry and 0 in any other entry. It is not difficult to check that $A_{j_0}^{2s}\sigma(A_{j_0}^{2s})^\dagger$ is the required matrix. Moreover, one can verify that:

$$\sum_{2s} A_{j_0}^{2s}\sigma(A_{j_0}^{2s})^\dagger = \frac{1}{2^{nk-1}} \begin{pmatrix} 0 & 0 & 0 & 0 & \ldots \\ 0 & a_{j_0}\mathbf{y}_{j_0} & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & a_{j_0}\mathbf{y}_{j_0} & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

b) Taking into account that $1 = \sum_{\mathbf{y}\in D_k(\mathbf{x})} \mathbf{y} = \sum_{j=1}^{2^{nk}} \mathbf{y}_j$, we have that:

$$1 - \sum_{j=1}^{2^{nk}} a_j\mathbf{y}_j = \sum_{j=1}^{2^{nk}} \mathbf{y}_j - \sum_{j=1}^{2^{nk}} a_j\mathbf{y}_j = \sum_{j=1}^{2^{nk}} (1 - a_j)\mathbf{y}_j.$$

We want to place $(1 - a_{j_0})\mathbf{y}_{j_0}$ in the $(2s-1, 2s-1)$-th entries of a $2^{nk} \times 2^{nk}$ matrix. Let us consider the $2^{nk} \times 2^{nk}$ matrix $A_{j_0}^{2s-1} = \sqrt{\frac{1-a_{j_0}}{2^{nk-1}}} D_{j_0}^{2s-1}$ such that $D_{j_0}^{2s-1}$ have 1 just in the $(2s-1, j_0)$-th entry and 0 in any other entry.
It is not difficult to check that $A_{j_0}^{2s-1}\sigma(A_{j_0}^{2s-1})^\dagger$ is the required matrix. Moreover, one can verify that:

$$\sum_{2s-1} A_{j_0}^{2s-1}\sigma(A_{j_0}^{2s-1})^\dagger =$$

$$= \frac{1}{2^{nk-1}} \begin{pmatrix} (1 - a_{j_0})\mathbf{y}_{j_0} & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & (1 - a_{j_0})\mathbf{y}_{j_0} & 0 & \ldots \\ 0 & 0 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Thus, we have that $\mathcal{P} = \sum_{j_0}\sum_{2s} A_{j_0}^{2s}\sigma(A_{j_0}^{2s})^\dagger + \sum_{j_0}\sum_{2s-1} A_{j_0}^{2s-1}\sigma(A_{j_0}^{2s-1})^\dagger =$

$$= (\frac{1}{2^{kn-1}} \otimes^{nk-1} I) \otimes \begin{pmatrix} 1 - \sum_{j=1}^{2^{nk}} a_j\mathbf{y}_j & 0 \\ 0 & \sum_{j=1}^{2^{nk}} a_j\mathbf{y}_j \end{pmatrix}.$$

Let us consider $A = \sum_{j_0}\sum_{2s}(A_{j_0}^{2s})^\dagger A_{j_0}^{2s} + \sum_{j_0}\sum_{2s+1}(A_{j_0}^{2s+1})^\dagger A_{j_0}^{2s+1}$. Our task is now to verify that $A = I$.

c) First of all, notice that the matrix $(A_{j_0}^{2s})^\dagger A_{j_0}^{2s}$ has the value $\frac{a_{j_0}}{2^{nk-1}}$ just in the (j_0, j_0)-th entry and 0 in any other entry. Therefore, the matrix $\sum_{2s}(A_{j_0}^{2s})^\dagger A_{j_0}^{2s}$ has the value $\frac{2^{nk-1}a_{j_0}}{2^{nk-1}} = a_{j_0}$ in the (j_0, j_0)-th entry and all the other entries are equal to 0. Hence:

$$\sum_{j_0}\sum_{2s}(A_{j_0}^{2s})^\dagger A_{j_0}^{2s} = \begin{pmatrix} a_1 & 0 & 0 & 0 & \ldots \\ 0 & a_2 & 0 & 0 & \ldots \\ 0 & 0 & a_3 & 0 & \ldots \\ 0 & 0 & 0 & a_4 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

d) On the other hand the matrix $(A_{j_0}^{2s-1})^\dagger A_{j_0}^{2s-1}$ has the value $\frac{1-a_{j_0}}{2^{nk-1}}$ just in the (j_0, j_0)-th entry and 0 in any other entry. Therefore, the matrix $\sum_{2s-1}(A_{j_0}^{2s-1})^\dagger A_{j_0}^{2s-1}$ has the value $\frac{2^{nk-1}(1-a_{j_0})}{2^{nk-1}} = 1 - a_{j_0}$ in the (j_0, j_0)-th entry and all the other entries are equal to 0. Hence:

$$\sum_{j_0}\sum_{2s-1}(A_{j_0}^{2s-1})^\dagger A_{j_0}^{2s-1} =$$

$$= \begin{pmatrix} 1-a_1 & 0 & 0 & 0 & \dots \\ 0 & 1-a_2 & 0 & 0 & \dots \\ 0 & 0 & 1-a_3 & 0 & \dots \\ 0 & 0 & 0 & 1-a_4 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Thus $\sum_{j_0}\sum_{2s}(A_{j_0}^{2s})^\dagger A_{j_0}^{2s-1} + \sum_{j_0}\sum_{2s-1}(A_{j_0}^{2s-1})^\dagger A_{j_0}^{2s-1} = I$ and $\mathcal{P}$ is a quantum operation.

■

IV. Representing the standard PMV-operations

The *standard PMV-algebra* [6], [10] is the algebra $[0,1]_{PMV} = \langle [0,1], \oplus, \bullet, \neg, 0, 1\rangle$, where $[0,1]$ is the real unit segment, $x \oplus y = \min(1, x+y)$, $\bullet$ is the usual product, and $\neg x = 1 - x$. This structure plays a notable role in quantum computing, in that it decribes, in a probabilistic way, a relevant system of quantum gates named *Poincarè irreversible quantum computational algebra* [2], [4].

Of course, $\neg$ can be given as a polynomial in the generator system $D_1(x)$, whence by Theorem 3.1, it is representable as a polynomial quantum operation. A possible representation can be the following: $NOT(\rho) = \sigma_x \rho \sigma_x^\dagger$. It is worth noting that $\mathtt{p}(NOT(\rho)) = 1 - \mathtt{p}(\rho)$.

Furthermore, $\bullet$ can be represented by a polynomial in the generator system $D_2(x,y)$. According with the construction presented in Theorem 3.1, the following representation obtains.
Let us consider the following matrices:

$$G_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad G_2 = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$G_3 = \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad G_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$G_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad G_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$G_7 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \quad G_8 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

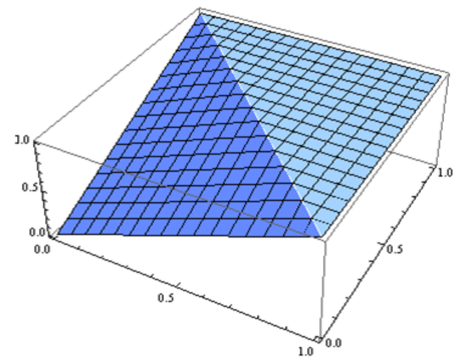

Fig. 1. The Łukasiewicz conorm

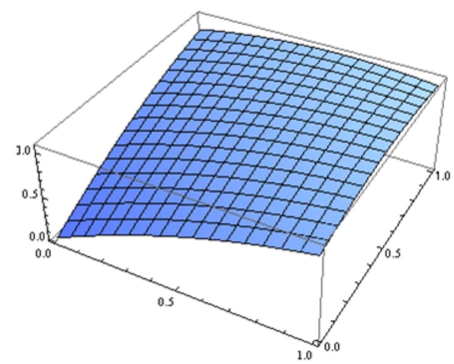

Fig. 2. $P(x,y)$

It is straightforward to check that $\sum_{i=1}^8 G_i(\tau\otimes\sigma)G_i^\dagger = \frac{1}{2} I \otimes \rho_{\mathtt{p}(\tau)\mathtt{p}(\sigma)}$ where $\sigma, \tau \in \mathcal{D}(\mathbb{C}^2)$.

Thus, by Kraus representation Theorem [9], $\sum_{i=1}^8 G_i(\tau\otimes\sigma)G_i^\dagger$ is a quantum operation, and $\mathtt{p}(\sum_{i=1}^8 G_i(\tau\otimes\sigma)G_i^\dagger) = \mathtt{p}(\tau)\bullet\mathtt{p}(\sigma)$. This quantum operation represents the well known quantum gate $IAND$ modulo a tensor power [2], [11].

Actually, the Łukasiewicz conorm $\oplus$ is not a polynomial, Figure 1.

Therefore, our idea is to give a polynomial $P(x,y)$ in some generator system $D_k(x,y)$, such that $P(x,y)$ can approximate the Łukasiewicz sum. By using numerical methods, we obtain the following polynomial approximant of $\oplus$ in $[0,1]$:

$$P(x,y) = \frac{5}{12}x(1-x) + \frac{5}{12}y(1-x) + \frac{5}{12}x(1-y) + \frac{5}{12}y(1-y) + \frac{1}{2}x + \frac{1}{2}y,$$

whose graph is depicted in Figure 2.

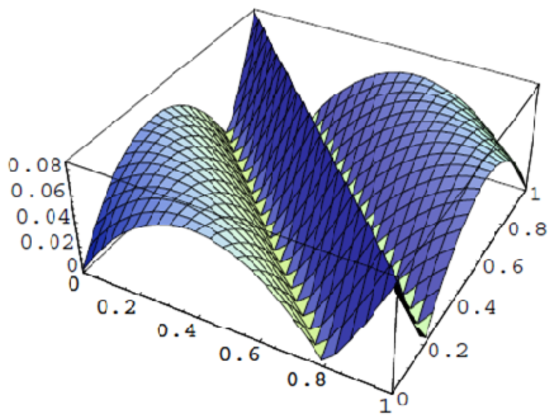

Fig. 3. $(x \oplus y) - P(x, y)$

Let us remark that $0 \leq P(x, y) \leq x \oplus y$. Therefore, $e = \max_{[0,1]^2}\{(x \oplus y) - P(x, y)\} \leq 0.08$ (see also Figure 3).

It can be seen that $P(x, y)$ is a polynomial given by the generator system $D_2(x, y)$, and it satisfies also the hypothesis of Theorem 3.1. Thus, $P(x, y)$ is representable as a polynomial quantum operation $\mathcal{P}_{\oplus}$, where $\mathtt{p}(\mathcal{P}_{\oplus}(\tau \otimes \sigma)) = (\mathtt{p}(\tau) \oplus \mathtt{p}(\sigma)) \pm 0.08$.

V. Conclusion

We can conclude that the accuracy of the obtained approximation is extremely high. This result provides a strong quantum computational motivation for the investigation of algebraic structures equipped with the Łukasiewicz sum and build a bridge between "classical" fuzzy logic and quantum computational logics.

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