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Representing continuous *t*-norms in quantum computation with mixed states

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Abstract

A model of quantum computation is discussed in (Aharanov *et al* 1997 *Proc. 13th Annual ACM Symp. on Theory of Computation, STOC* pp 20–30) and (Tarasov 2002 *J. Phys. A: Math. Gen.* **35** 5207–35) in which quantum gates are represented by quantum operations acting on mixed states. It allows one to use a quantum computational model in which connectives of a four-valued logic can be realized as quantum gates. In this model, we give a representation of certain functions, known as *t*-norms (Menger 1942 *Proc. Natl Acad. Sci. USA* **37** 57–60), that generalize the triangle inequality for the probability distribution-valued metrics. As a consequence an interpretation of the standard operations associated with the basic fuzzy logic (Hájek 1998 Metamathematics of Fuzzy Logic (*Trends in Logic* vol 4) (Dordrecht: Kluwer)) is provided in the frame of quantum computation.

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(Some figures in this article are in colour only in the electronic version)

Introduction

In the usual representation of quantum computational processes, a quantum circuit is identified with an appropriate composition of *quantum gates*, mathematically represented by *unitary operators* acting on pure states of a convenient (*n*-fold tensor product) Hilbert space $\otimes^n \mathbb{C}^2$ [30], and only takes into account reversible processes. But for many reasons this restriction is unduly. On the one hand, it does not encompass realistic physical states described by mixtures. In fact, a quantum system rarely is in a pure state. This may be caused, for example, by the in-complete efficiency in the preparation procedure and also by manipulations on the system as measurements over pure states, both of which produce statistical mixtures. Besides, systems cannot be completely isolated from the environment, undergoing decoherence of their states. Non-pure states, namely *mixed states*, are described by *density operators*. On the other hand, there are interesting processes that cannot be encoded in unitary evolutions, as measurements in the middle of the process. Several authors [1, 2, 6, 15, 32] have paid attention to a more general model of quantum computational processes, where pure states and unitary operators are replaced by density operators and quantum operations, respectively. In this case, time evolution is no longer necessarily reversible. In this powerful model we shall give a probabilistic-type representation for a family of functions known as continuous triangular norms.

Triangular norms (t-norms for short) where introduced by Menger [25] in the framework of the probabilistic metric spaces. These spaces may have an important application in quantum particle theory, particularly in connection with both string and $\epsilon^{(\infty)}$ theory [8]. In these spaces, t-norms allow us to generalize the triangle inequality for the probability distribution-valued metrics. They also play an important role in other areas such as decision making [9, 13], statistics [29] or cooperative games [3]. Formally, a t-norm is a binary operation $\odot$ on the real interval [0, 1] that satisfies the following:

(1) $x \odot 1 = x$,
(2) $x_1 \leqslant x_2, y_1 \leqslant y_2 \Longrightarrow x_1 \odot y_1 \leqslant x_2 \odot y_2$,
(3) $x \odot y = y \odot x$,
(4) $x \odot (y \odot z) = (x \odot y) \odot z$.

Since the 1980s, the interest in many-valued logics has increased enormously. In particular, the logics with truth values in [0, 1] emerged as a consequence of the 1965 proposal of fuzzy set theory by Zadeh [33] called *fuzzy logic*. In these logics, triangular norms were naturally proposed as interpretations of the conjunction. Seminal results of Hájek [16] have justified this choice. In fact, in the mentioned monograph, a system of fuzzy logic called *basic fuzzy logic* or *logic of continuous t-norms* is developed where it is possible to establish a genuine relationship between conjunction and implication; in this system the continuity of the t-norm plays an important role. The following are the three basic continuous t-norms:

(1) $x \odot_P y = x \cdot y$, (Product t-norm)
(2) $x \odot_Ł y = \max\{x + y - 1, 0\}$, (Łukasiewicz t-norm)
(3) $x \odot_G y = \min\{x, y\}$. (Gödel t-norm).

These t-norms are remarkable since each possible continuous t-norm can be obtained as an 'adequate combination' of the mentioned three [10, 21].

Our aim in this paper is to contribute to this research trend by focusing on a probabilistic-type representation of the three basic continuous t-norms in the model of quantum computation with mixed states [1].

This would allow us to establish a connection between quantum computation with mixed states and the fuzzy logic of continuous t-norms, thus providing a physical justification to several algebraic structures related to quantum computational logic with mixed states, such as *quasi-MV-algebras* [22, 23, 28], *quasi-PMV-algebras* [7], etc [11, 14, 18]. In fact, quantum computational logics with mixed states [5, 7, 15] may be presented as a logic $\langle \mathrm{Term}, \models \rangle$, where Term is an absolute free algebra (i.e. a language), whose natural universe of interpretation is a set $\mathcal{D}$ of density operators and whose connectives are naturally interpreted as certain quantum gates (i.e. quantum operations in the model of quantum computation with mixed states), which are formally introduced in section 2. More precisely, *canonical interpretations* are $Term$-homomorphisms $e : Term \to \mathcal{D}$. To define a relation of the semantic consequence '$\models$' based on the probability assignment, it is necessary to introduce the notion of canonical valuations. In fact, *canonical valuations* are functions over the unitary real interval $f : Term \to [0, 1]$

such that f can be factorized in the following way:

$$\begin{tikzcd} Term \arrow[r, "f"] \arrow[d, "e"'] \arrow[dr, phantom, "\equiv"] & {[0,1]} \\ \mathcal{D} \arrow[ur, "\mathtt{p}"'] & \end{tikzcd}$$

where $\mathtt{p}$ is a probability function obtained via the Born rule, also to be made precise in section 2. We will refer to these diagrams as *probabilistic semantics* [17, 31]. Then the semantical consequence $\models$ related to $\mathcal{D}$ is given by

$$\alpha \models \varphi \quad \text{iff} \quad \mathcal{R}[f(\alpha), f(\varphi)],$$

where $\mathcal{R}$ provides a relation between $f(\alpha)$ and $f(\varphi)$. An interesting case is the semantical consequence based on probability values equal to 1, more precisely, $\alpha \models \varphi$ iff $p(\alpha) = 1$ implies $p(\varphi) = 1$.

The connection between quantum computational logic with mixed states and fuzzy logic comes from the election of a system of quantum gates such that, when interpreted under probabilistic semantics, they turn out in some kind of operation in the real interval [0, 1] associated with fuzzy logic, as the continuous *t*-norms. The above-mentioned quasi-MV-algebras and quasi-PMV-algebras are some of the structures which represent algebraic counterparts of the probabilistic semantics conceived from the continuous *t*-norm.

On the other hand, the use of fuzzy logics (and infinite-valued Łukasiewicz logic in particular) in game theory and theoretical physics was pioneered in [26, 27], linking the mentioned structures with Ulam games and $AF - C^*$-algebras, respectively. We will pay special attention to the study of the Łukasiewicz *t*-norm due to its relation with Ulam games and its possible applications to error-correcting codes [24] in the context of quantum computation.

The paper is organized as follows: in section 1 we recall some basic notions about the model of quantum computer based on mixed states and quantum operations. In section 2 we introduce the concept of polynomial quantum operation and we establish a Stone–Weierstrass-type theorem for quantum operations. This provides a general framework for a probabilistic-type representation of continuous functions in the real interval [0, 1] as quantum operations in the sense of [19]. In section 3 we show an explicit representation as polynomial quantum operation for the product *t*-norm. In section 4, we give a representation for the Łukasiewicz and Gödel *t*-norms, approximating it by polynomial quantum operations and analyzing the error of the approximant; moreover, we study the efficiency of the method and, by use of numerical arguments, we provide more efficient representations.

1. Basic notions

A *quantum bit* or *qbit*, the fundamental concept of quantum computation, is a pure state in the Hilbert space $\mathbb{C}^2$. The standard orthonormal basis $\{|0\rangle, |1\rangle\}$ of $\mathbb{C}^2$, where $|0\rangle = (1, 0)$ and $|1\rangle = (0, 1)$, is sometimes called *logical basis*. This name refers to the fact that truth is related to $|1\rangle$ and falsity to $|0\rangle$. Thus, the pure states $|\psi\rangle$ in $\mathbb{C}^2$ are coherent superpositions of the basis vectors with complex coefficients: $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$ where $|c_0|^2 + |c_1|^2 = 1$. Quantum mechanics reads out the information content of a pure state via the Born rule. By these means, the qbit probability value we will be interested in is $\mathtt{p}(|\psi\rangle) = |c_1|^2$ corresponding to the basis vector associated with truth. In the usual representation of quantum computational processes, a quantum circuit is identified with an appropriate composition of *quantum gates*, mathematically represented by *unitary operators* acting on pure states of a convenient (*n*-fold tensor product) Hilbert space $\otimes^n\mathbb{C}^2$ [30]. In other words, the standard model for quantum computation is mathematically founded on '*qbit-unitary operators*'. In what follows we give a short description of the powerful model for quantum computers.

In terms of density matrices, a pure state $|\psi\rangle$ can be represented as a matrix product $\rho = |\psi\rangle\langle\psi|$. As a particular case, we may associate with each vector of the logical basis of $\mathbb{C}^2$ two density operators $P_0 = |0\rangle\langle 0|$ and $P_1 = |1\rangle\langle 1|$ that represent the falsity property and the truth property respectively in this framework. Due to the fact that the Pauli matrices:

$$\sigma_0 = I, \qquad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where $I = I^{(2)}$ is the 2×2 identity matrix, are a basis for the set of operators on $\mathbb{C}^2$, an arbitrary density operator ρ for n-qbits may be represented in terms of tensor products of them as $\rho = \frac{1}{2^n}\sum_{\mu_1,\ldots,\mu_n} a_{\mu_1,\ldots,\mu_n}(\sigma_{\mu_1} \otimes \cdots \otimes \sigma_{\mu_n})$ where $\mu_i \in \{0, x, y, z\}$ for each $i = 1,\ldots,n$ and $|a_{\mu_1,\ldots,\mu_n}| \leqslant 1$. We denote by $\mathcal{D}(\otimes^n\mathbb{C}^2)$ the set of all density operators on $\otimes^n\mathbb{C}^2$. Let us consider the operator $P_1^{(n)} = \otimes^{n-1} I \otimes P_1$ on $\otimes^n\mathbb{C}^2$. By applying the Born rule, we consider the probability of a density operator $\rho \in \mathcal{D}(\otimes^n\mathbb{C}^2)$ as follows:

$$\mathrm{p}(\rho) = Tr\big(P_1^{(n)}\rho\big).$$

Note that, in the particular case in which $\rho = |\psi\rangle\langle\psi|$ where $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$, we obtain that $\mathrm{p}(\rho) = |c_1|^2$. Thus, this probability value associated with ρ is the generalization, in this model, of the probability value considered for q-bits.

A *quantum operation* [19] is a linear operator $\mathcal{E} : \mathcal{L}(H_1) \to \mathcal{L}(H_2)$ where $\mathcal{L}(H_i)$ is the space of linear operators in the complex Hilbert space H_i ($i = 1, 2$), representable as $\mathcal{E}(\rho) = \sum_i A_i\rho A_i^\dagger$ where A_i are the operators satisfying $\sum_i A_i^\dagger A_i = I$ (Kraus representation). It can be seen that a quantum operation maps density operators into density operators. The new model '*density operators–quantum operations*' also called '*quantum computation with mixed states*' [1, 32] is equivalent in computational power to the standard one but gives a place to irreversible processes as measurements in the middle of the computation.

2. Polynomial quantum operations

We first introduce some notations and preliminary definitions. The term *multi-index* denotes an ordered n-tuple $\alpha = (\alpha_1,\ldots,\alpha_n)$ of non-negative integers α_i. If k is a natural number, $\alpha \leqslant k$ means that $\alpha_i \leqslant k$ for each $i \in \{1,\ldots,n\}$. The *order* of α is given by $|\alpha| = \alpha_1 + \cdots + \alpha_n$. If $\mathbf{x} = (x_1,\ldots,x_n)$ is an n-tuple of variables and $\alpha = (\alpha_1,\ldots,\alpha_n)$ is a multi-index, the monomial $\mathbf{x}^\alpha$ is defined by $\mathbf{x}^\alpha = x_1^{\alpha_1}x_2^{\alpha_2}\ldots x_n^{\alpha_n}$. In this language a polynomial of order k is a function $P(\mathbf{x}) = \sum_{|\alpha|\leqslant k} a_\alpha\mathbf{x}^\alpha$ such that $a_\alpha \in \mathbb{R}$.

Motivated by the Stone–Weierstrass theorem, Bernstein polynomials are considered by Lorentz [23] who proves that any continuous function $f(x)$ can be approximated by Bernstein polynomials on the real interval [0, 1]. We are interested in multivariate Bernstein polynomials. Let $\mathbf{x} = (x_1,\ldots,x_n)$, k be a natural number and $\alpha = (\alpha_1,\ldots,\alpha_n)$ be a multi-index such that $\alpha \leqslant k$. Then the Bernstein polynomial $B_{k,\alpha}(\mathbf{x})$ is defined as

$$B_{k,\alpha}(\mathbf{x}) = \prod_{i=1}^{n}\binom{k}{\alpha_i}(1-x_i)^{k-\alpha_i}x_i^{\alpha_i}.$$

Theorem 2.1 ([2, 20]). *Let* $\mathbf{x} = (x_1,\ldots,x_n)$ *and k be a positive integer. For any continuous function* $f : [0,1]^n \to \mathbb{R}$ *the polynomials*

$$B_k(f,\mathbf{x}) = \sum_{\alpha\leqslant k} f\left(\frac{\alpha_1}{k},\ldots,\frac{\alpha_n}{k}\right)B_{k,\alpha}(\mathbf{x})$$

converge to $f(\mathbf{x})$ *uniformly on* $[0,1]^n$ *when* $k \to \infty$. □

Let $\mathbf{x} = (x_1, \ldots, x_n)$ and k be a natural number. We first introduce the set $D_k(\mathbf{x})$ defined as

$$D_k(\mathbf{x}) = \left\{(1-x_1)^{\alpha_1} x_1^{\beta_1} \ldots (1-x_n)^{\alpha_n} x_n^{\beta_n} : \alpha_i + \beta_i = k, \quad i \in \{1, \ldots, n\}\right\}.$$

Let $\rho \in \mathcal{D}(\mathbb{C}^2)$ such that $\rho = \frac{1}{2}(I + r_x\sigma_x + r_y\sigma_y + r_z\sigma_z)$. Then it is clear that

$$\rho = \frac{1}{2}\begin{pmatrix} 1+r_z & r_x - \mathrm{i}r_y \\ r_x + \mathrm{i}r_y & 1 - r_z \end{pmatrix} = \begin{pmatrix} 1-\alpha & \beta \\ \beta^* & \alpha \end{pmatrix}.$$

Interestingly enough, any real number λ $(0 \leqslant \lambda \leqslant 1)$ uniquely determines a density operator

$$\rho_\lambda = (1-\lambda)P_0 + \lambda P_1 = \frac{1}{2}(I + (1-2\lambda)\sigma_z) = \begin{pmatrix} 1-\lambda & 0 \\ 0 & \lambda \end{pmatrix}.$$

It is easy to see that, if $\rho \in \mathcal{D}(\mathbb{C}^2)$, then $\mathtt{p}(\rho) = \frac{1-r_z}{2}$ and $\mathtt{p}(\rho_\lambda) = \lambda$. Thus each density operator ρ in $\mathcal{D}(\mathbb{C}^2)$ can be written as

$$\rho = \begin{pmatrix} 1-\mathtt{p}(\rho) & \beta \\ \beta^* & \mathtt{p}(\rho) \end{pmatrix}.$$

Proposition 2.2. *Let* $\mathbf{X}_1, \ldots, \mathbf{X}_n$ *be a family of matrices such that*

$$\mathbf{X}_i = \begin{pmatrix} 1-x_i & b_i \\ b_i^* & x_i \end{pmatrix}$$

and let us consider a tensor product $\mathbf{X} = (\otimes^k \mathbf{X}_1) \otimes (\otimes^k \mathbf{X}_2) \otimes \cdots \otimes (\otimes^k \mathbf{X}_n)$. *Then we have*

$$\mathrm{Diag}(\mathbf{X}) = D_k(x_1, \ldots, x_n)$$

where $\mathrm{Diag}(\mathbf{X})$ *denotes the set containing the diagonal entries of* $\mathbf{X}$.

Proof. By induction on k we can prove that $\mathrm{Diag}(\bigotimes^k X_i) = \{h_1 h_2 \ldots h_k : h_j \in \{(1-x_i), x_i\}, 1 \leqslant j \leqslant k\} = \{(1-x_1)^\alpha x_1^\beta : \alpha + \beta = k\}$. Thus, $\mathrm{Diag}((\otimes^k \mathbf{X}_1) \otimes (\otimes^k \mathbf{X}_2) \otimes \cdots \otimes (\otimes^k \mathbf{X}_n)) = \{(1-x_1)^{\alpha_1} x_1^{\beta_1} \ldots (1-x_n)^{\alpha_n} x_n^{\beta_n} : \alpha_i + \beta_i = k, \quad i \in \{1, \ldots, n\}\}$. Whence our claim follows. □

Proposition 2.3. *Let* $\mathbf{x} = (x_1, \ldots, x_n)$ *and* k *be a natural number. Given any monomial* $\mathbf{x}^\alpha$ *such that* $|\alpha| \leqslant k$, *there exist coefficients* $\delta_\mathbf{y}$:

$$\mathbf{x}^\alpha = \sum_{\mathbf{y} \in D_k(\mathbf{x})} \delta_\mathbf{y} \mathbf{y} \qquad \text{and} \qquad 1 - \mathbf{x}^\alpha = \sum_{\mathbf{y} \in D_k(\mathbf{x})} (1 - \delta_\mathbf{y})\mathbf{y},$$

where $\delta_\mathbf{y} \in \{0, 1\}$.

Proof. For each $i \in \{1, \ldots, n\}$, consider the matrix $\mathbf{X_i}$ given by

$$\mathbf{X}_i = \begin{pmatrix} 1-x_i & 0 \\ 0 & x_i \end{pmatrix}.$$

Let $\mathbf{x}^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$ such that $|\alpha| \leqslant k$. Thus, there exist $s_1, \ldots, s_n$ such that $\alpha_i + s_i = k$. Let $\mathbf{W} = (\otimes^{s_1} \mathbf{X}_1) \otimes (\otimes^{s_2} \mathbf{X}_2) \otimes \cdots \otimes (\otimes^{s_n} \mathbf{X}_n)$ and let us consider the matrix $\mathbf{W}\mathbf{x}^\alpha$. In view of lemma 2.2, $\mathrm{Diag}(\mathbf{W}\mathbf{x}^\alpha) \subseteq D_k(x_1, \ldots, x_n)$ since every element in $\mathrm{Diag}(\mathbf{W}\mathbf{x}^\alpha)$ is a monomial of order nk. Further, since $\mathrm{tr}(\mathbf{W}\mathbf{x}^\alpha) = (\mathrm{tr}\mathbf{W})\mathbf{x}^\alpha = 1\mathbf{x}^\alpha = \mathbf{x}^\alpha$, we have that $\mathbf{x}^\alpha = \mathrm{tr}(\mathbf{W}\mathbf{x}^\alpha)$ is the required polynomial expansion.

For the case $1 - \mathbf{x}^\alpha$, consider the matrix $\mathbf{X} = (\otimes^k \mathbf{X}_1) \otimes (\otimes^k \mathbf{X}_2) \otimes \cdots \otimes (\otimes^k \mathbf{X}_n)$. By lemma 2.2 $\mathrm{Diag}(\mathbf{X}) = D_k(x_1, \ldots, x_n)$ and $\mathrm{tr}(\mathbf{X}) = 1$. Taking into account that

$\mathbf{x}^\alpha = \sum_{\mathbf{y}\in D_k(\mathbf{x})} \delta_\mathbf{y}\mathbf{y}$ we have that $1 = \mathrm{tr}(\mathbf{X}) = \sum_{\mathbf{y}\in D_k(\mathbf{x})} \delta_\mathbf{y}\mathbf{y} + \sum_{\mathbf{y}\in D_k(\mathbf{x})}(1-\delta_\mathbf{y})\mathbf{y} = \mathbf{x}^\alpha + \sum_{\mathbf{y}\in D_k(\mathbf{x})}(1-\delta_\mathbf{y})\mathbf{y}$. Thus $1 - \mathbf{x}^\alpha = \sum_{\mathbf{y}\in D_k(\mathbf{x})} \gamma_y \mathbf{y}$. □

Definition 2.4. *A quantum operation $\mathcal{P} : \mathcal{L}(\otimes^{nk}\mathbb{C}^2) \to \mathcal{L}(\otimes^{nk}\mathbb{C}^2)$ is called the polynomial quantum operation iff there exists a polynomial $P(x_1,\ldots,x_n)$ such that for each n-tuple $(\sigma_1,\ldots,\sigma_n)$ in $\mathcal{D}(\mathbb{C}^2)$ we have that*

$$\mathrm{p}(\mathcal{P}((\otimes^k\sigma_1)\otimes\cdots\otimes(\otimes^k\sigma_n))) = P(\mathrm{p}(\sigma_1),\ldots,\mathrm{p}(\sigma_n)).$$

Theorem 2.5. *Let $\mathbf{x} = (x_1,\ldots,x_n)$ be an n-tuple of variables and consider the set $D_k(\mathbf{x})$. Let $P(\mathbf{x}) = \sum_{\mathbf{y}\in D_k(\mathbf{x})} a_\mathbf{y}\mathbf{y}$ be a polynomial such that $\mathbf{y}\in D_k(\mathbf{x})$, $0\leqslant a_\mathbf{y}$ and $0 \leqslant P(\mathbf{x})\mid_{[0,1]^n} \leqslant 1$. Consider $M = \max_{\mathbf{y}\in D_k(\mathbf{x})}\{a_\mathbf{y}\}$ and define*

$$k_M = \begin{cases} k, & \text{if} \quad M \leqslant 1 \\ \min z : M \leqslant 2^{nz-1}, & \text{if} \quad M > 1. \end{cases} \tag{1}$$

Then there exists a polynomial quantum operation $\mathcal{P} : \mathcal{L}(\otimes^{nk_M}\mathbb{C}^2) \to \mathcal{L}(\otimes^{nk_M}\mathbb{C}^2)$ such that for each n-tuple $\sigma = (\sigma_1,\ldots,\sigma_n)$ in $\mathcal{D}(\mathbb{C}^2)$

$$\mathrm{p}(\mathcal{P}((\otimes^{k_M}\sigma_1)\otimes\cdots\otimes(\otimes^{k_M}\sigma_n))) = P(\mathrm{p}(\sigma_M),\ldots,\mathrm{p}(\sigma_n)).$$

Moreover, $\mathcal{P}((\otimes^k\sigma_1)\otimes\cdots\otimes(\otimes^k\sigma_n)) = \left(\frac{1}{2^{nk_M-1}}\otimes^{nk_M-1} I\right)\otimes\rho_{P(\mathrm{p}(\sigma_1),\ldots,\mathrm{p}(\sigma_n))}$.

Proof. By lemma 2.3 we can consider the polynomial $P(\mathbf{x})$ in the generator system $D_{k_M}(\mathbf{x})$, i.e. $P(\mathbf{x}) = \sum_{\mathbf{y}\in D_{k_M}(\mathbf{x})} a_\mathbf{y}\mathbf{y}$. Let $P_M(\mathbf{x})$ be the polynomial given by

$$P_M(\mathbf{x}) = \frac{1}{2^{nk_M-1}}P(\mathbf{x}) = \sum_{\mathbf{y}\in D_{k_M}(\mathbf{x})}\frac{a_\mathbf{y}}{2^{nk_M-1}}\mathbf{y} = \sum_{\mathbf{y}\in D_{k_M}(\mathbf{x})} b_\mathbf{y}\mathbf{y}.$$

By the definition of k_M it is clear that $0\leqslant b_\mathbf{y}\leqslant 1$ for each $\mathbf{y}\in D_{k_M}(\mathbf{x})$. Let $\sigma_1,\ldots,\sigma_n$ be the density operators of $\mathbb{C}^2$. Assume that for any σ_i

$$\sigma_i = \begin{pmatrix} 1-x_i & b_i \\ b_i^* & x_i \end{pmatrix}.$$

Hence, $\mathrm{p}(\sigma_i) = x_i$. It is clear that $\sigma = (\otimes^{k_M}\sigma_1)\otimes\cdots\otimes(\otimes^{k_M}\sigma_n)$ is a matrix of order $2^{nk_M}\times 2^{nk_M}$ and, by lemma 2.2, $\mathrm{Diag}(\sigma) = D_{k_M}(x_1,\ldots,x_n)$. Thus, each $\mathbf{y}\in D_{k_M}(\mathbf{x})$ can be seen as the (i,i)th entry of $\mathrm{Diag}(\sigma)$. Further, the polynomial $P_M(\mathbf{x}) = \sum_{\mathbf{y}\in D_{k_M}(\mathbf{x})} b_\mathbf{y}\mathbf{y} = \sum_{j=1}^{2^{nk_M}} b_j\mathbf{y}_j$ is such that every $\mathbf{y}_j$ is the (j,j)th entry of $\mathrm{Diag}(\sigma)$. Let, now, $\mathbf{y}_{j_0}\in\mathrm{Diag}(\sigma)$.

(a) We want to place $b_{j_0}\mathbf{y}_{j_0}$ in the $(2s,2s)$th entries of a $2^{nk_M}\times 2^{nk_M}$ matrix, for any $s\geqslant 1$. Let us consider the $2^{nk_M}\times 2^{nk_M}$ matrix $A_{j_0}^{2s} = \sqrt{b_{j_0}}D_{j_0}^{2s}$ such that $D_{j_0}^{2s}$ has 1 just in the $(2s, j_0)$th entries and 0 in any other entry. It is not difficult to check that $A_{j_0}^{2s}\sigma\left(A_{j_0}^{2s}\right)^\dagger$ is the required matrix. Moreover, one can verify that

$$\sum_{2s} A_{j_0}^{2s}\sigma\left(A_{j_0}^{2s}\right)^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & b_{j_0}\mathbf{y}_{j_0} & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & b_{j_0}\mathbf{y}_{j_0} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

(b) Taking into account that $1 = \sum_{\mathbf{y}\in D_{k_M}(\mathbf{x})}\mathbf{y} = \sum_{j=1}^{2^{nk_M}}\mathbf{y}_j$, we have that

$$\frac{1}{2^{nk_M-1}} - \sum_{j=1}^{2^{nk_M}} b_j\mathbf{y}_j = \sum_{j=1}^{2^{nk_M}}\frac{\mathbf{y}_j}{2^{nk_M-1}} - \sum_{j=1}^{2^{nk_M}} b_j\mathbf{y}_j = \sum_{j=1}^{2^{nk_M}}\left(\frac{1}{2^{nk_M-1}} - b_j\right)\mathbf{y}_j.$$

By the definition of b_j it is clear that $\frac{1}{2^{nk_M-1}} \geqslant b_j$.

We want to place $\left(\frac{1}{2^{nk_M-1}} - b_{j_0}\right)\mathbf{y}_{j_0}$ in the $(2s-1, 2s-1)$st entries of a $2^{nk_M} \times 2^{nk_M}$ matrix, for all $s \geqslant 1$. Let us consider the $2^{nk_M} \times 2^{nk_M}$ matrix $A_{j_0}^{2s-1} = \sqrt{\frac{1}{2^{nk_M-1}} - b_{j_0}} D_{j_0}^{2s-1}$ such that $D_{j_0}^{2s-1}$ have 1 just in the $(2s-1, j_0)$th entries and 0 in any other entry. It is not difficult to check that $A_{j_0}^{2s-1}\sigma(A_{j_0}^{2s-1})^\dagger$ is the required matrix. Moreover, one can verify that

$$\sum_{2s-1} A_{j_0}^{2s-1}\sigma(A_{j_0}^{2s-1})^\dagger = \begin{pmatrix} \left(\frac{1}{2^{nk_M-1}} - b_{j_0}\right)\mathbf{y}_{j_0} & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & \left(\frac{1}{2^{nk_M-1}} - b_{j_0}\right)\mathbf{y}_{j_0} & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Thus, we have that $\mathcal{P}(\sigma) = \sum_{j_0}\sum_{2s} A_{j_0}^{2s}\sigma\left(A_{j_0}^{2s}\right)^\dagger + \sum_{j_0}\sum_{2s-1} A_{j_0}^{2s-1}\sigma\left(A_{j_0}^{2s-1}\right)^\dagger$

$$= (\otimes^{nk-1} I) \otimes \begin{pmatrix} \frac{1}{2^{nk_M-1}} - \sum_{j=1}^{2^{nk_M}} b_j\mathbf{y}_j & 0 \\ 0 & \sum_{j=1}^{2^{nk_M}} b_j\mathbf{y}_j \end{pmatrix}.$$

Let us consider $A = \sum_{j_0}\sum_{2s}\left(A_{j_0}^{2s}\right)^\dagger A_{j_0}^{2s} + \sum_{j_0}\sum_{2s+1}(A_{j_0}^{2s+1})^\dagger A_{j_0}^{2s+1}$. Our task is now to verify that $A = I$.

(c) First of all, note that the matrix $\left(A_{j_0}^{2s}\right)^\dagger A_{j_0}^{2s}$ has the value b_{j_0} just in the (j_0, j_0)th entry and 0 in any other entry. Therefore, the matrix $\sum_{2s}\left(A_{j_0}^{2s}\right)^\dagger A_{j_0}^{2s}$ has the value $2^{nk_M-1}b_{j_0}$ in the (j_0, j_0)th entry and all the other entries are equal to 0. Hence,

$$\sum_{j_0}\sum_{2s}\left(A_{j_0}^{2s}\right)^\dagger A_{j_0}^{2s} = \begin{pmatrix} 2^{nk_M-1}b_1 & 0 & 0 & \dots \\ 0 & 2^{nk_M-1}b_2 & 0 & \dots \\ 0 & 0 & 2^{nk_M-1}b_3 & \dots \\ & & & \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

(d) On the other hand, the matrix $\left(A_{j_0}^{2s-1}\right)^\dagger A_{j_0}^{2s-1}$ has the value $\frac{1}{2^{nk_M-1}} - b_{j_0}$ just in the (j_0, j_0)th entry and 0 in any other entry. Therefore, the matrix $\sum_{2s-1}\left(A_{j_0}^{2s-1}\right)^\dagger A_{j_0}^{2s-1}$ has the value $2^{nk_M-1}\left(\frac{1}{2^{nk_M-1}} - b_{j_0}\right) = 1 - 2^{nk_M-1}b_{j_0}$ in the (j_0, j_0)th entry and all the other entries are equal to 0. Hence,

$$\sum_{j_0}\sum_{2s-1}\left(A_{j_0}^{2s-1}\right)^\dagger A_{j_0}^{2s-1} = \begin{pmatrix} 1 - 2^{nk_M-1}b_1 & 0 & 0 & \dots \\ 0 & 1 - 2^{nk_M-1}b_2 & 0 & \dots \\ 0 & 0 & 1 - 2^{nk_M-1}b_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Thus $\sum_{j_0}\sum_{2s}\left(A_{j_0}^{2s}\right)^\dagger A_{j_0}^{2s-1} + \sum_{j_0}\sum_{2s-1}(A_{j_0}^{2s-1})^\dagger A_{j_0}^{2s-1} = I$ and $\mathcal{P}$ is a quantum operation. Taking into account the matrix representation of $\mathcal{P}$ in $\sigma = (\otimes^{k_1}\sigma_1) \otimes \cdots \otimes (\otimes^{k_M}\sigma_n)$, we have that $\mathrm{p}(\mathcal{P}(\sigma)) = \mathrm{tr}(P_1\mathcal{P}(\sigma)) = 2^{nk_M-1}P_M(\mathbf{x}) = P(\mathbf{x})$ as is required. □

Theorem 2.6. *Let $f : [0,1]^n \to [0,1]$ be a continuous function. Then for each $\epsilon > 0$ there exists a quantum operation $\mathcal{P}_\epsilon : \mathcal{L}(\otimes^{nk}\mathbb{C}^2) \to \mathcal{L}(\otimes^{nk}\mathbb{C}^2)$ such that for each $\sigma = (\sigma_1, \dots, \sigma_n)$ in $\mathcal{D}(\mathbb{C}^2)$,*

$$|\mathrm{p}(\mathcal{P}_\epsilon((\otimes^k\sigma_1) \otimes \cdots \otimes (\otimes^k\sigma_n))) - f(\mathrm{p}(\sigma_1)\dots\mathrm{p}(\sigma_n))| \leqslant \epsilon.$$

Proof. Suppose that $0 < \epsilon \leqslant 1$. Consider the function $(1 - \frac{\epsilon}{2})f$. By theorem 2.1, there exists a natural number $k_{\frac{\epsilon}{2}}$ such that $|B_{k_{\frac{\epsilon}{2}}}(f, \mathbf{x}) - (1 - \frac{\epsilon}{2})f(\mathbf{x})| \leqslant \frac{\epsilon}{2}$ for $\mathbf{x}$ in $[0, 1]^n$. Consequently, and taking into account that the coefficient of $B_{k_{\frac{\epsilon}{2}}}(f, \mathbf{x})$ is positive, we can see that $0 \leqslant B_{k_{\frac{\epsilon}{2}}}((1-\frac{\epsilon}{2})f, \mathbf{x}) \leqslant 1$ for all $\mathbf{x}$ in $[0, 1]^n$. It is not very hard to see that $B_{k_{\frac{\epsilon}{2}}}((1-\frac{\epsilon}{2})f, \mathbf{x})$ can be represented in the generator system $D_k(\mathbf{x})$ where $k = nk_{\frac{\epsilon}{2}}$. Thus by theorem 2.5 there exists a quantum operation $\mathcal{P}_\epsilon : \mathcal{L}(\otimes^{nk}\mathbb{C}^2) \to \mathcal{L}(\otimes^{nk}\mathbb{C}^2)$ such that

$$\mathrm{p}(\mathcal{P}_\epsilon((\otimes^k\sigma_1) \otimes \cdots \otimes (\otimes^k\sigma_n))) = B_{k_{\frac{\epsilon}{2}}}\left(\left(1 - \frac{\epsilon}{2}\right) f, \mathrm{p}(\sigma_1) \ldots \mathrm{p}(\sigma_n)\right).$$

On the other hand, $|B_{k_{\frac{\epsilon}{2}}}((1 - \frac{\epsilon}{2})f, \mathbf{x}) - f(\mathbf{x})| \leqslant |B_{k_{\frac{\epsilon}{2}}}((1 - \frac{\epsilon}{2})f, \mathbf{x}) - (1 - \frac{\epsilon}{2})f(\mathbf{x})| + \left|(1 - \frac{\epsilon}{2})f(\mathbf{x}) - f(\mathbf{x})\right| = \frac{\epsilon}{2} + \left|\frac{\epsilon}{2}f(\mathbf{x})\right| \leqslant \epsilon$ in $[0, 1]^n$. □

Remark 2.7. It is clear that theorem 2.6 is a kind of Stone–Weierstrass theorem since it allows one to represent up to an ϵ any possible continuous function $f : [0, 1]^n \to [0, 1]$ by means of quantum operations. However, it is inefficient to implement this result for a large n in view of the fact that it requires many copies of the involved states. In order to use this kind of representation, thus, the key idea is finding a polynomial with a low degree which can approximate the function itself minimizing the error.

3. Representing a product *t*-norm

Note that the product *t*-norm is given by a polynomial in the generator system $D_2(x, y)$ satisfying the hypothesis of theorem 2.5. Thus it is representable as a polynomial quantum operation. More precisely, it is not very hard to show this representation. In fact, let us consider the following matrices:

$$G_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad G_2 = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad G_3 = \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$G_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad G_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad G_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$G_7 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \qquad G_8 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is straightforward to check that $\sum_{i=1}^{8} G_i(\tau \otimes \sigma)G_i^\dagger = \frac{1}{2}I \otimes \rho_{\mathrm{p}(\tau)\mathrm{p}(\sigma)}$ where $\sigma, \tau \in \mathcal{D}(\mathbb{C}^2)$. Thus $\mathrm{p}(\sum_{i=1}^{8} G_i(\tau \otimes \sigma)G_i^\dagger) = \mathrm{p}(\tau) \cdot \mathrm{p}(\sigma)$ and

$$\tau \odot_P \sigma = \sum_{i=1}^{8} G_i(\tau \otimes \sigma)G_i^\dagger$$

provide a representation of the product *t*-norm as a polynomial quantum gate.

4. Representing Łukasiewicz and Gödel *t*-norms

Since the Łukasiewicz *t*-norm is not a polynomial, it is impossible to give a representation of such in terms of polynomial quantum operations. But this function is a continuous function in $[0, 1]$; hence, it can be uniformly approximated by a polynomial quantum operation in the sense of theorem 2.6. Bernstein approximants of $x \odot_Ł y$ are $p_n(x, y) = \sum_{(\alpha_1,\alpha_2)\leqslant k} (\frac{\alpha_1}{k} \odot_Ł \frac{\alpha_2}{k}) B_{k,(\alpha_1,\alpha_2)}(x, y)$. Our proposal is to give an explicit form of the Bernstein approximants and the error. For this we study another function in $[0, 1]$ related to the Łukasiewicz *t*-norm called *Łukasiewicz sum* or *Łukasiewicz t-conorm*, introduced as dual operation of the Łukasiewicz *t*-norm, and we obtain the Bernstein approximants of this function by using another argument. This method provides an estimation error in the approximation. The Łukasiewicz sum is defined as the De Morgan dual of $\odot$:

$$x \oplus y = 1 - ((1 - x) \odot_Ł (1 - y)). \tag{2}$$

Observe that in the standard model of the infinite-valued Łukasiewicz logic $x \oplus y = \min(1, x + y)$ and $x \odot_Ł y = 1 - ((1 - x) \oplus (1 - y))$.

To explicitly construct a family of approximant polynomials we introduce the auxiliary function $[0, 2] \ni z \mapsto g(z) = \min(1, z)$. As $x \oplus y = g(x + y)$, the problem of approximating the bivariate function $\oplus$ is changed into the easier problem of approximating the one-variable function $g(z)$ in $[0, 2]$. We observe that, by defining

$$[0, 2] \ni z \mapsto h(z) = \begin{cases} \frac{z}{2}, & \text{if } z \in [0, 1], \\ 1 - \frac{z}{2}, & \text{if } z \in (1, 2], \end{cases}$$

we have $g(z) = \frac{z}{2} + h(z)$ and $h(z)$ is symmetric with respect to the point $z = 1$, i.e. $h(2 - z) = h(z)$. For this reason we approximate $h(z)$ using the symmetric functions $z^i(2 - z)^i$. We look for some of the coefficients c_i such that

$$\frac{z}{2} = \sum_{i=1}^{\infty} c_i z^i (2 - z)^i, \qquad z \in [0, 1]. \tag{3}$$

If such coefficients exist and are positive, then the sequence of partial sums

$$h_n(z) = \sum_{i=1}^{n} c_i z^i (2 - z)^i, \qquad z \in [0, 2], \tag{4}$$

is monotonic and, by Dini's theorem, it converges uniformly to $h(z)$. It follows that $g_n(z) = \frac{z}{2} + h_n(z)$ is monotonic, increasing with respect to n, and it converges uniformly to $g(z)$. The coefficients c_i can be obtained by resorting to the binomial series. Indeed, the change of variable $w = z(2 - z)$ in $[0, 1]$ shows that equation (3) can be rewritten as $\frac{1-\sqrt{1-w}}{2} = \sum_i c_i w^i$, where $w \in [0, 1]$. The binomial series for the square root, i.e. $\sqrt{1 - w} = \sum_{i=0}^{\infty} (-1)^i \binom{1/2}{i} x^i$, uniformly convergent if $|w| \leqslant 1$, allows us to obtain $c_0 = 0$ and $c_i = \frac{(-1)^{i+1}}{2} \binom{1/2}{i} > 0$ for $i > 0$. Thus we can explicitly describe the approximant $G_n(x, y) = g_n(x + y)$ of $x \oplus y$ as

$$G_n(x, y) = \frac{x + y}{2} + \sum_{i=1}^{n} \frac{(-1)^{i+1}}{2} \binom{1/2}{\mathrm{i}} (x + y)^i ((1 - x) + (1 - y))^i$$

and it can be interpreted in terms of Bernstein polynomials.

We now briefly obtain a bound for the approximation error

$$e_n = \max_{x,y \in [0,1]} |(x \oplus y) - G_n(x, y)|.$$

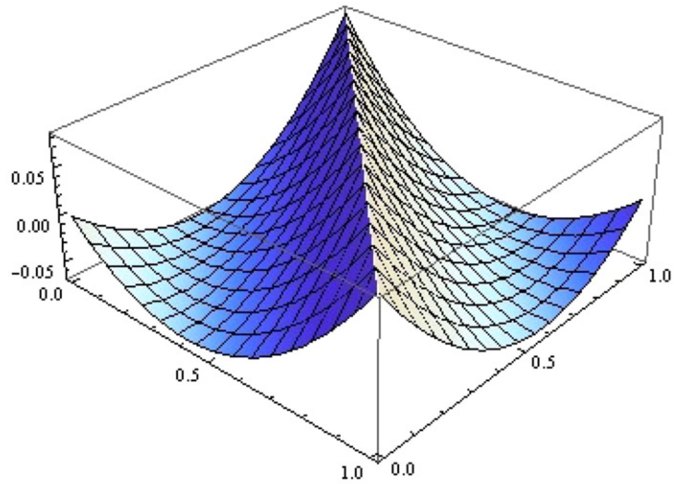

Figure 1. Errors in the approximations l_1, i.e. $(x \oplus y) - l_1(x, y)$, for $x, y \in [0, 1]$.

Indeed, it can be proved that $0 \leqslant g(z) - g_n(z) \leqslant g(1) - g_n(1)$ when $z \in [0, 1]$, and e_n is maximum when $x + y = 1$, so we can write

$$e_n = 1 - g_n(1) = \sum_{i=n+1}^{\infty} c_i = 2(n+1)c_{n+1} = \frac{1}{2\sqrt{\pi n}} + O(n^{-3/2}).$$

To obtain the approximant for $x \odot_Ł y$ we first consider the polynomial $l_n(x, y) = 1 - g_n((1 - x) + (1 - y))$. Note that $|(x \odot_Ł y) - (1 - g_n((1 - x) + (1 - y)))| = x \odot_Ł y) - (1 - g_n((1 - x) + (1 - y))) = e_n(x, y)$, i.e. the error is the same. Now we consider a polynomial quantum operation $\mathcal{G}_n$ associated with $g_n((1 - x) + (1 - y))$ and we define the following composition of quantum operations:

$$\mathcal{L}_n = \sigma_x^n \mathcal{G}_n\left(\sigma_x^n\right)^\dagger,$$

where $\sigma_x^n = (\otimes^{n-2} I) \otimes \sigma_x$. With a similar argument we can exhibit a quantum operation for the Gödel t-norm since $\min\{x, y\} = x \odot_Ł ((1 - x) \oplus y)$.

Taking into account remark 2.7 we construct the approximant in $D_2(x, y)$ and $D_4(x, y)$ for the Łukasiewicz sum.

Case $D_2(x, y)$. Our Bernstein approximant for $n = 1$, i.e. $G_1(x, y)$, provides the error $2(n+1)c_{n+1} = \frac{1}{4}$. Using numerical methods we can obtain a better approximation in $D_2(x, y)$. In fact, by considering the following polynomial:

$$l_1(x, y) = \tfrac{5}{12}(x + y)(1 - x) + \tfrac{5}{12}(x + y)(1 - y) + \tfrac{1}{2}(x + y),$$

we have that it approximates the Łukasiewicz sum with a maximum error bound of 0.08 as show in figure 1.

Case $D_4(x, y)$. Our Bernstein approximant for $n = 2$, i.e. $G_1(x, y)$, provides the error $2(n + 1)c_{n+1} = \frac{3}{16}$. Using numerical methods again, we can obtain the following polynomial in the same generator system $D_4(x, y)$:

$$\begin{aligned} l_2(x, y) = {} & \tfrac{1485}{2970}(x + y) + \tfrac{21}{2970}(x + y)((1 - x) + (1 - y)) \\ & + \tfrac{1372}{2970}(x + y)^2((1 - x)^2 + 2(1 - x)(1 - y) + (1 - y)^2) \end{aligned}$$

with a maximum error 0.04 as shown in figure 2.

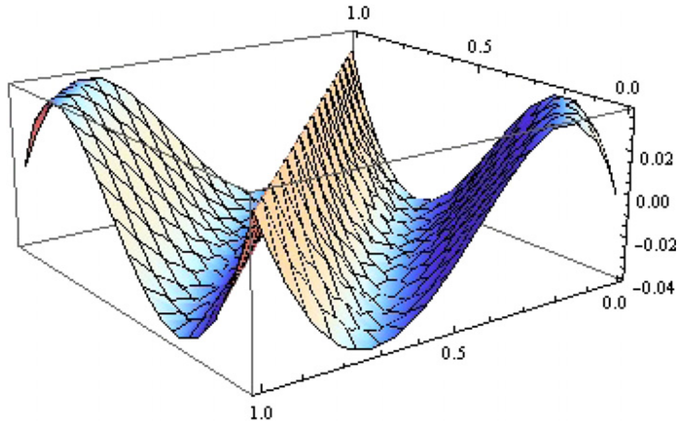

Figure 2. Errors in the approximations l_2, i.e. $(x \oplus y) - l_2(x, y)$, for $x, y \in [0, 1]$.

This allows one to represent the standard operations associated with the Basic fuzzy logic [16] as quantum operations, providing a physical interpretation to several systems of quantum computational logic [7, 12, 15].

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