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Entanglement as a Semantic Resource

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Abstract The characteristic holistic features of the quantum theoretic formalism and the intriguing notion of *entanglement* can be applied to a field that is far from microphysics: logical semantics. *Quantum computational logics* are new forms of quantum logic that have been suggested by the theory of quantum logical gates in quantum computation. In the standard semantics of these logics, sentences denote quantum information quantities: systems of qubits (*quregisters*) or, more generally, mixtures of quregisters (*qumixes*), while logical connectives are interpreted as special *quantum logical gates* (which have a characteristic *reversible* and *dynamic* behavior). In this framework, states of knowledge may be entangled, in such a way that our information about the whole determines our information about the parts; and the procedure cannot be, generally, inverted. In spite of its appealing properties, the standard version of the quantum computational semantics is strongly "Hilbert-space dependent". This

Dedicated to Peter Mittelstaedt.

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certainly represents a shortcoming for all applications, where real and complex numbers do not generally play any significant role (as happens, for instance, in the case of natural and of artistic languages). We propose an abstract version of quantum computational semantics, where abstract qumixes, quregisters and registers are identified with some special objects (not necessarily living in a Hilbert space), while gates are reversible functions that transform qumixes into qumixes. In this framework, one can give an abstract definition of the notions of *superposition* and of *entangled pieces of information*, quite independently of any numerical values. We investigate three different forms of *abstract holistic quantum computational logic*.

Keywords Quantum computation · Entanglement · Holistic semantics

1 Introduction

There is something paradoxical in the history of entanglement-phenomena and of the EPR-situations. As is well known, in the Thirties, the notion of entanglement had been often described as one of the basic mysteries that cause the strange behavior of quantum objects. Later on, however, most of the features that had been represented as negative consequences of the quantum formalism have been converted into theoretic and practical advantages, even from the technological point of view. We need only think that *quantum computation*, *teleportation* and *quantum cryptography* systematically use, in a positive way, some characteristic quantum phenomena that are essentially connected with entanglement-situations.

A less investigated application of the notion of entanglement concerns a field that is apparently far from microphysics: logical semantics. Interestingly enough, logicians are now beginning to understand how the enigmatic quantum entanglement can be also applied to a formal analysis of some characteristic semantic phenomena, where *holistic* and *contextual* aspects play a relevant role. As is well known, the traditional semantic theories, based on classical logic, are essentially *analytical* and *antiholistic*. A basic principle in these theories is a *compositionality-assumption*, according to which the meaning of any compound expression is determined by the meanings of its parts. Furthermore, meanings are always supposed to be *precise* and *non-ambiguous*. As a consequence, classical semantics turns out to hardly applicable to an adequate formal analysis either of natural languages or of the languages of art, where contextuality and ambiguity seem to represent essential features. We need only think how difficult is dealing with contextual meanings in the framework of computer-translations! The quantum-theoretic formalism, instead, gives rise to *entangled states of knowledge* where our information about the *whole* determines our information about the *parts*. And the procedure cannot be, generally, inverted: in other words, it is impossible to reconstruct the global information as a mere combination of the partial pieces of information about the component elements. Once broken into its parts, the puzzle cannot be put back together!

Quantum computation has recently suggested some new forms of quantum logic that have been called *quantum computational logics* (**QCL**'**s**). The basic semantic idea underlying these logics can be described as a natural generalization of classical logic. Let us refer to an "information-theoretic" presentation of classical semantics (and of classical circuit theory). In this framework, sentences are supposed to denote classical bits (either 1 or 0), while the Boolean connectives (*not*, *and*, *or*) represent *logical gates*: functions that permit us to process information. The sentences of **QCL′s**, instead, are supposed to represent quantum pieces of information that are generally uncertain: systems of *qubits* (also called *quregisters*), or more generally *mixtures* of quregisters (also called *qumixes*). At the same time, the logical connectives are interpreted as *quantum logical gates*.

What are exactly quantum logical gates? As is well known, in quantum theory the dynamic evolution of quantum objects is governed by Schrödinger-equation. Accordingly, for any times t_0 and t_1, a pure state $|\psi(t_0)\rangle$ of an object at time t_0 is transformed into another pure state of the same object at time t_1 by means of a unitary operator U (which represents a reversible transformation):

$$|\psi(t_1)\rangle = U(|\psi(t_0)\rangle).$$

Quantum logical gates (briefly, *gates*) are special examples of unitary operators that transform quregisters into quregisters in a reversible way. Hence, from an intuitive point of view, the application of a sequence of gates to an input-quregister can be regarded as the dynamic evolution of a quantum object that is processing a given amount of quantum information. By definition gates are unitary operators whose domains consist of vectors of convenient Hilbert spaces. However they can be naturally generalized also to qumixes.

As is well known, in any semantic characterization of a given logic, the basic concept is represented by the notion of *model* (or *interpretation*) of the language. For instance, in classical semantics a *model* of a sentential language is a homomorphic map that assigns a classical bit to any sentence, by interpreting the logical connectives as the corresponding Boolean functions.

In the holistic semantics of **QCL′s** a *model* (or *interpretation*) of the language is a map $\mathtt{Hol}$ that assigns to any sentence α a qumix that represents the informational meaning of α:

$$\alpha \mapsto \mathtt{Hol}(\alpha).$$

As expected, any model $\mathtt{Hol}$ shall preserve the logical form of the sentences, by interpreting any connective $\circ$ of the language as a corresponding gate G°. Furthermore, the qumix $\mathtt{Hol}(\alpha)$ should live in a Hilbert space whose dimension depends on the logical form of α. The simplest examples of sentences are *atomic* sentences (which cannot be decomposed into more elementary sentences). Accordingly, the meaning of such sentences shall live in the simplest Hilbert space: the two-dimensional space $\mathbb{C}^2$ (where all qubits are located). A molecular sentence with n occurrences of atomic sentences can be regarded as a linguistic description of a compound physical system consisting of n particles. In fact, we need n particles in order to carry the information that is expressed by our molecular sentence. On this basis, it is natural to assume that the meaning of such a sentence lives in the n-fold tensor product of $\mathbb{C}^2$.

The holistic features of our semantics depend on the fact that any model $\mathtt{Hol}$ assigns to any sentence α a *global* meaning, that cannot be generally inferred from the meanings assigned by $\mathtt{Hol}$ to the atomic parts of α. What happens here is just the opposite with respect to the standard behavior of compositional semantics: $\mathtt{Hol}(\alpha)$ determines the meanings of all its parts, which turn out to be essentially *context-dependent*. As a consequence, any sentence may receive different meanings in different contexts. Going from the *whole* to the *parts* is here possible because all logical operations are reversible: one can go back and forth without any dissipation of information!

A fundamental role in this semantic game is played the notion of *entanglement*, which is mathematically based on the characteristic properties of tensor products. As is well known, from an intuitive point of view the basic features of an *entangled state* $|\psi\rangle$ can be sketched as follows:

- $|\psi\rangle$ is a maximal information (a pure state) that describes a compound physical system S (say, a two-electron system);
- the information determined by $|\psi\rangle$ about the parts of S is non-maximal. Hence, the states of the whole system is a pure state, while the states of the parts (which are determined by the state of the whole and are usually called *reduced states*) are proper mixtures. It may also happen that the state of the compound system (although representing a maximum of information) describes the parts as essentially indiscernible objects, that cannot satisfy any characteristic individual property. One obtains, in this way, an apparent violation of Leibniz' indiscernibility principle.

Entanglement-phenomena can be naturally used to model some typical holistic semantic situations in the framework of our quantum computational semantics. We can consider entangled quregisters that are meanings of molecular sentences. As an example, consider a conjunctive sentence having the form

$$\gamma = \alpha \wedge \beta.$$

The following situation is possible:

- the *meaning* $\mathtt{Hol}(\gamma)$ of the conjunction γ is a quregister, which represents a maximal information (a pure state);
- the meanings of the parts (α, β) are quantum-entangled and cannot be represented by two pure states (two quregisters).

We can say that the *sharp meaning* of the conjunction determines two *ambiguous meanings* for the parts (α, β), which are represented by two mixed states. In other words, *the meaning of the whole determines the meanings of the parts, but not the other way around*. In fact, one cannot go back from the two ambiguous meanings of the parts to the quregister representing the meaning of the whole. The mixed state representing the ambiguous meaning of α (of β) can be regarded as a kind of *contextual meaning* of α (of β), determined by the *global context*, which corresponds to the quregister $\mathtt{Hol}(\alpha \wedge \beta)$ (the meaning of the conjunction $\alpha \wedge \beta$).

In spite of its appealing properties, the standard version of the quantum computational semantics is strongly "Hilbert-space dependent". This certainly represents a shortcoming for all applications, where real and complex numbers do not generally play any significant role (as happens, for instance, in the case of natural and of artistic languages).

Is it sensible to look for an *abstract quantum computational semantics*? We will positively answer this question, by defining a notion of *abstract quantum computational structure*, where abstract qumixes, quregisters and registers are identified with some special objects (not necessarily living in a Hilbert space), while gates are reversible functions that transform qumixes into qumixes. From an intuitive point of view, abstract qumixes and quregisters represent pieces of information that are generally *uncertain*, while (abstract) registers are special examples of quregisters that store a *certain* information. In this framework, one can give an abstract definition of the notions of *superposition* and of *entangled pieces of information*, quite independently of any numerical values.

As expected, one can show that concrete (Hilbert-space) qumix-structures are special examples of abstract quantum computational structures.

2 Concrete and Abstract Quantum Computational Structures

Let us first recall some basic concepts of quantum computation that play an important logical role. Consider the two-dimensional Hilbert space $\mathbb{C}^2$ (where any vector is represented by a pair of complex numbers). Let $\mathcal{B}^{(1)} = \{|0\rangle, |1\rangle\}$ be the canonical orthonormal basis for $\mathbb{C}^2$ (where $|0\rangle = (1,0)$ and $|1\rangle = (0,1)$).

Definition 2.1 (Qubit) A *qubit* is a unit vector $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$ of the Hilbert space $\mathbb{C}^2$.

The basis-elements $|0\rangle$ and $|1\rangle$ represent, in this framework, the two classical bits, which can be also interpreted as the classical truth-values *false* and *true*, respectively. Hence, from an intuitive point of view any qubit $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$ can be regarded as a kind of "quantum perhaps": a superposition of the two classical truth-values, where the *Falsity* has probability $|c_0|^2$, while the *Truth* has probability $|c_1|^2$. From the physical point of view, a qubit describes the pure state of a single particle, while a system of n qubits (also called *n-quregister*) corresponds to the state of a compound system consisting of n particles. Accordingly, any n-quregister can be represented as a unit vector of the n-fold tensor product of the space $\mathbb{C}^2$:

$$\mathcal{H}^{(n)} := \underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{n\text{-times}},$$

(where $\mathcal{H}^{(1)} := \mathbb{C}^2$). We will use $x, y, \ldots$ as variables ranging over the set $\{0,1\}$, while $|x\rangle, |y\rangle, \ldots$ will range over the basis $\mathcal{B}^{(1)}$. Any factorized unit vector $|x_1\rangle \otimes \cdots \otimes |x_n\rangle$ of the space $\mathcal{H}^{(n)}$ will represent in this framework a *classical register* (a sequence of n bits). Instead of $|x_1\rangle \otimes \cdots \otimes |x_n\rangle$ we will also write $|x_1, \ldots, x_n\rangle$. The set $\mathcal{B}^{(n)}$ of all classical registers is an orthonormal basis for the space $\mathcal{H}^{(n)}$.

Quregisters are pure states, hence *maximal pieces of information*, that cannot be consistently extended to a richer knowledge. In quantum computation one cannot help referring also to non-maximal pieces of information; these correspond to *mixtures of quregisters* (also called *qumixes*), which are mathematically represented by density operators.

Definition 2.2 (Qumix) A *qumix* is a density operator of a Hilbert space $\mathcal{H}^{(n)}$.

We will indicate by $\mathfrak{D}(\mathcal{H}^{(n)})$ the set of all qumixes of $\mathcal{H}^{(n)}$, while $\mathfrak{D}^{\mathcal{H}} := \bigcup_{n=1}^{\infty}(\mathcal{H}^{(n)})$ will denote the set of all possible qumixes.

Of course, quregisters can be described as special cases of qumixes.

The algebraic structure of the set $\mathfrak{D}$ of all qumixes essentially depends on the definition of a probability-function $\mathtt{p}$ that assigns to any qumix ρ a probability-value. From an intuitive point of view, $\mathtt{p}(\rho)$ represents the probability that the quantum information stored by ρ corresponds to a *true* information. In order to define the function $\mathtt{p}$, we will first identify in any space $\mathcal{H}^{(n)}$ two special projections $P_0^{(n)}$ and $P_1^{(n)}$ that will represent the *Falsity* and the *Truth* properties, respectively. In this way, *Falsity* and *Truth* are dealt with as special cases of physical properties to which any density operator assigns a well determined probability-value, according to the quantum theoretic formalism.

Let us first distinguish in any space $\mathcal{H}^{(n)}$ the *true* from the *false* registers:

$$|x_1, \ldots, x_n\rangle \text{ is called } \textit{true} \text{ iff } x_n = 1;$$

$$|x_1, \ldots, x_n\rangle \text{ is called } \textit{false} \text{ iff } x_n = 0.$$

In other words, the last bit of a given register determines its truth-value. On this basis, the truth-property of the space $\mathcal{H}^{(n)}$ can be naturally identified with the projection-operator $P_1^{(n)}$ whose range is the closed subspace spanned by the set of all true registers (hence, $P_1^{(n)}$ transforms every vector into a vector that is a superposition of true registers). Dually, the falsity-property of $\mathcal{H}^{(n)}$ is identified with the projection-operator $P_0^{(n)}$ whose range is the closed subspace spanned by the set of all false registers. Following the Born-rule we can now define the probability that the information ρ satisfies the truth-property as follows:

$$\mathtt{p}(\rho) := \mathtt{Tr}(\rho P_1^{(n)}), \quad \text{where } \mathtt{Tr} \text{ is the trace functional.}$$

As an example, suppose we are in the simplest situation where our qumix ρ corresponds to a single qubit, having the standard form:

$$|\psi\rangle = c_0|0\rangle + c_1|1\rangle.$$

In such a case, we obtain: $\mathtt{p}(\rho) = |c_1|^2$.

Let us now turn to quantum logical gates. As is well known, the classical circuit-model of computation, both in its reversible and in its irreversible version, can be formulated by using a very small set of gates, called *universal set of gates*. This property (termed *functional universality*) amounts to saying that every gate can be mathematically simulated by means of a convenient composition of gates belonging to the universal set. For instance, in the irreversible case, the single gate $\mathtt{NAND}$ or the system consisting of the two gates $\mathtt{AND}$ and $\mathtt{NOT}$ turn out to be functionally universal. In the reversible case, such a role is played by a single gate: the *Toffoli gate* (also called *controlled-controlled not*). Since there are uncountably many unitary operators, there is no hope to find any *finite* functionally universal set of quantum gates.

The best we can do is having recourse to the notion of finite *approximate universality* [8]: a finite set of gates is said to be *approximately universal* iff any quantum gate can be approximated up to an arbitrary accuracy by a quantum circuit that consists of elements of this set. Apparently, finding simpler and simpler sets of universal gates represents a crucial step in order to try and realize concrete quantum computers.

An interesting gate-system that is approximately universal has been discovered by Shi [8] and Aharonov [1]. This system consists of two gates: the *Toffoli gate* and the *Hadamard gate* (also called the *squareroot of the identity*). From a foundational point of view, we can say that the Hadamard gate is just all that the Toffoli gate needs in order to reach quantum (approximate) universality, starting from classical (functional) universality.[1] This is the reason why it is interesting to investigate a formal semantics for quantum computational languages, whose basic connectives correspond to the Toffoli gate and to the Hadamard gate, respectively.

We will now define the two gates of the Shi-Aharonov system. The Toffoli gate represents the classical part of the system: a classically universal gate, that permits us to define the reversible versions of all Boolean functions.

Definition 2.3 (The Toffoli gate) For any $n, m, p \geq 1$, the *Toffoli gate* is the linear operator $\mathtt{T}^{(n,m,p)}$ defined on $\mathcal{H}^{(n+m+p)}$ such that, for every element $|x_1, \ldots, x_n\rangle \otimes |y_1, \ldots, y_m\rangle \otimes |z_1, \ldots, z_p\rangle$ of the basis $\mathcal{B}^{(n+m+p)}$,

$$\begin{aligned}&\mathtt{T}^{(n,m,p)}(|x_1, \ldots, x_n\rangle \otimes |y_1, \ldots, y_m\rangle \otimes |z_1, \ldots, z_p\rangle)\\&\quad = |x_1, \ldots, x_n\rangle \otimes |y_1, \ldots, y_m\rangle \otimes |z_1, \ldots, z_{p-1}\rangle \otimes |x_n y_m \widehat{+} z_p\rangle,\end{aligned}$$

where $\widehat{+}$ represents the sum modulo 2.

One can easily show that $\mathtt{T}^{(n,m,p)}$ is a unitary operator.

The Boolean functions $\mathtt{AND}$, $\mathtt{NAND}$, $\mathtt{NOT}$ can be now defined in terms of the Toffoli gate.

Definition 2.4

- For any $|\psi\rangle \in \mathcal{H}^{(n)}$ and for any $|\varphi\rangle \in \mathcal{H}^{(m)}$,

$$\mathtt{AND}(|\psi\rangle, |\varphi\rangle) := \mathtt{T}^{(n,m,1)}(|\psi\rangle \otimes |\varphi\rangle \otimes |0\rangle);$$

- For any $|\psi\rangle \in \mathcal{H}^{(n)}$ and for any $|\varphi\rangle \in \mathcal{H}^{(m)}$,

$$\mathtt{NAND}(|\psi\rangle, |\varphi\rangle) := \mathtt{T}^{(n,m,1)}(|\psi\rangle \otimes |\varphi\rangle \otimes |1\rangle);$$

- For any $|\psi\rangle \in \mathcal{H}^{(n)}$,

$$\mathtt{NOT}(|\psi\rangle) := \mathtt{T}(|1\rangle \otimes |1\rangle \otimes |\psi\rangle).$$

[1] The algebraic structure of the Shi-Aharonov system has been investigated in [3].

Defining the Boolean negation $\mathtt{NOT}$ in terms of the Toffoli gate has, however, a shortcoming that is determined by the increasing of the dimension of the Hilbert space. Namely, if $|\psi\rangle$ belongs to $\mathcal{H}^{(n)}$, then its negation $\mathtt{NOT}(|\psi\rangle)$ belongs to $\mathcal{H}^{(n+2)}$.

For computational and logical aims, the following independent definition of the negation-gate is more economical:

Definition 2.5 (The negation) For any $n \geq 1$, the *negation* on $\mathcal{H}^{(n)}$ is the linear operator $\mathtt{Not}^{(n)}$ such that, for every element $|x_1, \ldots, x_n\rangle$ of the basis $\mathcal{B}^{(n)}$,

$$\mathtt{Not}^{(n)}(|x_1, \ldots, x_n\rangle) = |x_1, \ldots, x_{n-1}\rangle \otimes |1 - x_n\rangle.$$

Accordingly, in the following, we will prefer to assume $\mathtt{Not}^{(n)}$ as a primitive gate.

The Toffoli gate is a classical (reversible) gate: whenever the input is a classical register, then also the output will be a classical register. In other words, the gate is incapable to "create" superpositions. The "genuine" quantum component of the Shi-Aharonov system is represented by the Hadamard gate (also called the *squareroot of the identity*).

Definition 2.6 (The squareroot of the identity) For any $n \geq 1$, the *squareroot of the identity* on $\mathcal{H}^{(n)}$ is the linear operator $\sqrt{\mathtt{I}}^{(n)}$ such that for every element $|x_1, \ldots, x_n\rangle$ of the basis $\mathcal{B}^{(n)}$:

$$\sqrt{\mathtt{I}}^{(n)}(|x_1, \ldots, x_n\rangle) = |x_1, \ldots, x_{n-1}\rangle \otimes \frac{1}{\sqrt{2}}((-1)^{x_n}|x_n\rangle + |1 - x_n\rangle).$$

The basic property of $\sqrt{\mathtt{I}}^{(n)}$ is the following:

$$\text{for any } |\psi\rangle \in \mathcal{H}^{(n)}, \quad \sqrt{\mathtt{I}}^{(n)}(\sqrt{\mathtt{I}}^{(n)}(|\psi\rangle)) = |\psi\rangle.$$

By definition, gates are unitary operators whose domains consist of vectors of convenient Hilbert spaces. At the same time, gates can be naturally generalized also to qumixes. Such generalizations that transform qumixes into qumixes in a reversible way, are called *qumix-gates* (or *unitary quantum operations* [2]). Suppose that $\mathtt{G}$ is a gate of $\mathcal{H}^{(n)}$. Then the corresponding qumix-gate ${}^{\mathcal{D}}\mathtt{G}$ is defined as follows:

$${}^{\mathcal{D}}\mathtt{G}(\rho) := \mathtt{G}\rho\mathtt{G}^*,$$

where ρ is a density operator of $\mathcal{H}^{(n)}$ and $\mathtt{G}^*$ is the adjoint of $\mathtt{G}$. Accordingly, ${}^{\mathcal{D}}\mathtt{Not}^{(n)}$, ${}^{\mathcal{D}}\mathtt{T}^{(m,n,p)}$ and ${}^{\mathcal{D}}\sqrt{\mathtt{I}}^{(n)}$ will represent the negation, the Toffoli and the Hadamard qumix-gates, respectively. At the same time, ${}^{\mathcal{D}}\mathtt{AND}(\rho, \sigma)$ and ${}^{\mathcal{D}}\mathtt{NAND}(\rho, \sigma)$ are supposed to be defined in the expected way.

The following theorems describe some basic properties of our qumix-gates.

Theorem 2.1 [7]

(1) $\mathtt{p}({}^{\mathcal{D}}\mathtt{Not}^{(n)}(\rho)) = 1 - \mathtt{p}(\rho)$;
(2) $\mathtt{p}({}^{\mathcal{D}}\mathtt{AND}(\rho, \sigma)) = \mathtt{p}(\rho)\mathtt{p}(\sigma)$;

(3) $\mathtt{p}({}^{\mathcal{D}}\mathtt{NAND}(\rho,\sigma)) = 1 - \mathtt{p}(\rho)\mathtt{p}(\sigma)$.

Theorem 2.2 [4]

(1) *For any* $\rho \in \mathfrak{D}(\mathcal{H}^{(n)})$: ${}^{\mathcal{D}}\sqrt{\mathtt{I}}^{(n)}({}^{\mathcal{D}}\sqrt{\mathtt{I}}^{(n)}(\rho)) = \rho$;
(2) $\forall n \in \mathbb{N}^+ : \mathtt{p}({}^{\mathcal{D}}\sqrt{\mathtt{I}}^{(n)}(k_n P_1^{(n)})) = \mathtt{p}({}^{\mathcal{D}}\sqrt{\mathtt{I}}^{(n)}(k_n P_0^{(n)})) = \frac{1}{2}$, *where* $k_n := \frac{1}{2^{n-1}}$.

Theorem 2.3 [3] *Let* $\rho \in \mathfrak{D}(\mathcal{H}^{(n)})$, $\sigma \in \mathfrak{D}(\mathcal{H}^{(m)})$ *and* $\tau \in \mathfrak{D}(\bigotimes^p \mathbb{C}^2)$. *Then,*

$$\mathtt{p}({}^{\mathcal{D}}\mathtt{T}^{(n,m,p)}(\rho \otimes \sigma \otimes \tau)) = (1 - \mathtt{p}(\tau))\mathtt{p}(\rho)\mathtt{p}(\sigma) + \mathtt{p}(\tau)(1 - \mathtt{p}(\rho)\mathtt{p}(\sigma)).$$

As a consequence of Theorem 2.3 and of Theorem 2.1, the probability-value $\mathtt{p}({}^{\mathcal{D}}\mathtt{T}^{(n,m,p)}(\rho \otimes \sigma \otimes \tau))$ can be regarded as a kind of weighted sum of $\mathtt{p}({}^{\mathcal{D}}\mathtt{AND}(\rho,\sigma))$ and of $\mathtt{p}({}^{\mathcal{D}}\mathtt{NAND}(\rho,\sigma))$, with weight $\mathtt{p}({}^{\mathcal{D}}\mathtt{Not}^{(p)}(\tau))$ and $\mathtt{p}(\tau)$, respectively.

The set $\mathfrak{D}^{\mathcal{H}}$ of all qumixes can be preordered by the relation $\preceq$ that is defined as follows in terms of the probability function $\mathtt{p}$.

Definition 2.7 (The qumix-preorder) For any qumixes $\rho \in \mathfrak{D}(\mathcal{H}^{(m)})$ and $\sigma \in \mathfrak{D}(\mathcal{H}^{(n)})$,

$$\rho \preceq \sigma \text{ iff } \mathtt{p}(\rho) \leq \mathtt{p}(\sigma) \quad \text{and} \quad \mathtt{p}\Big({}^{\mathcal{D}}\sqrt{\mathtt{I}}^{(m)}(\rho)\Big) \leq \mathtt{p}\Big({}^{\mathcal{D}}\sqrt{\mathtt{I}}^{(n)}(\sigma)\Big).$$

One can easily show that $\preceq$ is reflexive and transitive. This permits us to define, in the expected way, an equivalence relation $\equiv$ on the set $\mathfrak{D}^{\mathcal{H}}$.

Definition 2.8 $\rho \equiv \sigma$ iff $\rho \preceq \sigma$ and $\sigma \preceq \rho$.

One can prove that $\equiv$ is a congruence relation with respect to the gates Toffoli, negation and squareroot of identity.

We will now try and abstract from concrete (Hilbert-space) quantum computational structures by introducing the notion of *abstract quantum computational structure*. For the sake of simplicity, we will preserve the same terminology and the same notation that is normally used for Hilbert-space objects. As expected, some basic concepts that can be defined in the Hilbert-space framework have to be assumed as primitive in the abstract case.

From an intuitive point of view the basic features of an abstract quantum computational structure can be sketched as follows.

- The elements of the domain $\mathfrak{D}$, called *abstract qumixes*, represent abstract pieces of information that admit three levels of uncertainty, corresponding to three different subdomains of $\mathfrak{D}$:

  (1) the subdomain $\mathfrak{R}$ contains the *registers*, which represent classical certainties;
  (2) the subdomain $\mathcal{Q}$ contains the *quregisters*, which represent possibly uncertain pieces of information;

(3) the total domain $\mathfrak{D}$ also contains *mixtures* of quregisters, which represent pieces of information that are generally ambiguous.

We have: $\mathfrak{R} \subseteq \mathcal{Q} \subseteq \mathfrak{D}$.

- Any object ρ of $\mathfrak{D}$ has a well determined length n, which represents the complexity of the information stored by ρ. We will indicate by $\mathfrak{R}^{(n)}$, $\mathcal{Q}^{(n)}$, $\mathfrak{D}^{(n)}$, respectively, the set of the registers, of the quregisters, of the qumixes whose length is n.
- Unlike the concrete case, the abstract reduced state function is primitive. If ρ lives in the domain $\mathfrak{D}^{(n)}$ and consists of k parts, $Red^{(j)}(\rho)$ represents the j-th component of the global information ρ, for any $j \leq k$.
- For any $n \geq 1$, $\clubsuit^n$ and $\spadesuit^n$ are two binary relations (called *quconsistency* and *mixconsistency*) that may hold between qumixes of $\mathfrak{D}^{(n)}$. One is dealing with two different forms of consistency-relations, which can be defined in the Hilbert-space structures. Quconsisteny collapses into identity in the case of registers, while mixconsistency collapses into identity in the case of quregisters. Interestingly enough, quconsistency permits us to define an abstract notion of *superposition*, which is independent of any numerical values (superpositions without amplitudes!). Any quregister $|\psi\rangle$ is called a *superposition* of all registers that are quconsistent with $|\psi\rangle$. As a consequence, classical certainties (registers) turn out to be only superpositions of themselves.
- Also the abstract preorder relation $\preceq$ is primitive. As we have seen, in the concrete case, $\preceq$ is defined in terms of the probability function $\mathtt{p}$, which permits us to evaluate (in a quantitative way) the "relative distance" from the truth for any pair of qumixes ρ and σ. By preserving this intuitive idea, we suppose that in the abstract case $\rho \preceq \sigma$ means: σ is closer to the truth than ρ.
- The abstract versions of the gates are supposed to satisfy some minimal conditions that hold for the corresponding concrete gates.

Now we can give the precise definition of *abstract quantum computational structure*.

Definition 2.9 (Abstract quantum computational structure) An abstract quantum computational structure is a system

$$\mathfrak{A} = (\mathfrak{D}, Red, \clubsuit, \spadesuit, \preceq, \mathfrak{G}, \mathtt{Not}, \mathtt{I}, \sqrt{\mathtt{I}}, \mathtt{T}, |0\rangle, |1\rangle),$$

where the following conditions hold:

(1) $\mathfrak{D}$ is the set of all *abstract qumixes* (briefly, *qumixes*), indicated by $\rho, \sigma, \ldots$.
   (a) $\mathfrak{D} = \bigcup_{n\geq 1} \mathfrak{D}^{(n)}$, where $\mathfrak{D}^{(n)}$ is the set of all *qumixes of length n*, indicated by $\rho^{(n)}, \sigma^{(n)}, \ldots$.
   (b) The cartesian product $\mathfrak{D}^{(m)} \times \mathfrak{D}^{(n)}$ is embeddable into $\mathfrak{D}^{(m+n)}$. We indicate by $\rho^{(m)} \otimes \sigma^{(n)}$ the element of $\mathfrak{D}^{(m+n)}$ that corresponds to the pair $(\rho^{(m)}, \sigma^{(n)})$. We assume that: $\rho^{(m)} \otimes (\sigma^{(n)} \otimes \tau^{(p)}) = (\rho^{(m)} \otimes \sigma^{(n)}) \otimes \tau^{(p)}$.

(2) $\mathcal{Q}$ is the set of all *abstract quregisters* (briefly, *quregisters*), indicated by $|\psi\rangle$, $|\varphi\rangle, \ldots$.
   (a) $\mathcal{Q} = \bigcup_{n\geq 1} \mathcal{Q}^{(n)}$, where $\mathcal{Q}^{(n)}$ is the set of all *quregisters of length n*, indicated by $|\psi\rangle^{(n)}, |\varphi\rangle^{(n)}, \ldots$.

(b) The cartesian product $\mathcal{Q}^{(m)} \times \mathcal{Q}^{(n)}$ is embeddable into $\mathcal{Q}^{(m+n)}$. We indicate by $|\psi\rangle^{(m)} \otimes |\varphi\rangle^{(n)}$ the element of $\mathcal{Q}^{(m+n)}$ that corresponds to the pair $(|\psi\rangle^{(m)}, |\varphi\rangle^{(n)})$. We assume that: $|\psi\rangle^{(m)} \otimes (|\varphi\rangle^{(n)} \otimes |\chi\rangle^{(p)}) = (|\psi\rangle^{(m)} \otimes |\varphi\rangle^{(n)}) \otimes |\chi\rangle^{(p)}$.

(c) For any $n \geq 1$, $\mathcal{Q}^{(n)} \subseteq \mathfrak{D}^{(n)}$.

(3) For any $n \geq 1$, $\mathfrak{R}^{(n)}$ is the set of all *registers of length* n. The elements of $\mathfrak{R}^{(n)}$ are represented as sequences $|x_1, \ldots, x_n\rangle$, where $x_i \in \{0, 1\}$. The set $\mathfrak{R}^{(1)} = \{|0\rangle, |1\rangle\}$ is called the set of the two *abstract bits*.

(a) $\mathfrak{R}^{(n)} \subseteq \mathcal{Q}^{(n)}$;

(b) $\mathfrak{R}^{(m+n)}$ is in one-to-one correspondence with the cartesian product $\mathfrak{R}^{(m)} \times \mathfrak{R}^{(n)}$. We indicate by $|x_1, \ldots, x_m, y_1, \ldots, y_n\rangle$ the register in $\mathfrak{R}^{(m+n)}$ that corresponds to the pair $(|x_1, \ldots, x_m\rangle, |y_1, \ldots, y_n\rangle)$.

(4) *Red* is the *reduced information map*. Consider a qumix $\rho^{(m)}$ and let $m = n_1 + \ldots + n_k$. For any i s.t. $1 \leq i \leq k$, $Red^i(\rho^{(m)})$ is a qumix that satisfies the following conditions:

(a) $Red^i(\rho^{(m)}) \in \mathfrak{D}^{(n_i)}$;

(b) $\rho^{(m)} = \rho^{(n_1)} \otimes \cdots \otimes \rho^{(n_k)} \curvearrowright Red^i(\rho^{(m)}) = \rho^{(n_i)}$.

(5) $\clubsuit$ is a map that associates to any $n \geq 1$ a binary reflexive and symmetric relation $\clubsuit^n$ (called *quconsistency*) that may hold between qumixes of length n. The following conditions hold:

(a) $|x_1, \ldots, x_n\rangle \clubsuit^n |y_1, \ldots, y_n\rangle \curvearrowright |x_1, \ldots, x_n\rangle = |y_1, \ldots, y_n\rangle$;

(b) any qumix of length n is quconsistent with at least one register of length n.

Let $Reg(\rho^{(n)}) = \{|x_1, \ldots, x_n\rangle : |x_1, \ldots, x_n\rangle \clubsuit^n \rho^{(n)}\}$.

When $\rho^{(n)}$ is a quregister, we say that $\rho^{(n)}$ is a *superposition* of the elements of $Reg(\rho^{(n)})$.

(6) $\spadesuit$ is a map that associates to any $n \geq 1$ a binary reflexive and symmetric relation $\spadesuit^n$ (called *mixconsistency*) that may hold between qumixes of length n.

(a) $|\psi\rangle^{(n)} \spadesuit^n |\varphi\rangle^{(n)} \curvearrowright |\psi\rangle^{(n)} = |\varphi\rangle^{(n)}$.

(b) $\rho^{(n)} \spadesuit^n \sigma^{(n)} \curvearrowright \rho^{(n)} \clubsuit^n \sigma^{(n)}$.

Let $Mix(\rho^{(n)}) = \{|\psi\rangle^{(n)} : |\psi\rangle^{(n)} \spadesuit^n \rho^{(n)}\}$.

We say that $\rho^{(n)}$ is a *mixture* of the elements of $Mix(\rho^{(n)})$.

(7) $\preceq$ is a *preorder relation* on $\mathfrak{D}$.

This permits one to define the following equivalence relation:

$$\rho \approx \sigma := \rho \preceq \sigma \quad \text{and} \quad \sigma \preceq \rho.$$

The following conditions hold:

(a) $\rho^{(m)} \otimes \sigma^{(n)} \approx \sigma^{(n)}$;

(b) $\rho^{(m)} \otimes (\sigma^{(n)} \otimes \tau^{(p)}) \approx (\rho^{(m)} \otimes \sigma^{(n)}) \otimes \tau^{(p)}$;

(c) For any m s.t. $m = n_1 + \cdots + n_k$ and for any i s.t. $1 \leq i \leq k$, $\rho^{(m)} \approx Red^1(\rho^{(m)}) \otimes \cdots \otimes Red^k(\rho^{(m)})$.

(8) Registers satisfy the following conditions:

(a) $|x_1, \ldots, x_m\rangle \preceq |y_1, \ldots, y_n, 1\rangle$;

(b) $|x_1, \ldots, x_m, 0\rangle \preceq |y_1, \ldots, y_n\rangle$.

(9) $\mathfrak{G}$ is a map that assigns to any $n \geq 1$ the set $\mathfrak{G}^{(n)}$ of all *abstract gates* (briefly, *gates*) defined on $\mathfrak{D}^{(n)}$.

By *gate on* $\mathfrak{D}^{(n)}$ we mean a map $G^{(n)}$ that satisfies the following conditions:

(a) $G^{(n)}$ is an injection of $\mathfrak{D}^{(n)}$ into $\mathfrak{D}^{(n)}$ (this guarantees that gates are reversible logical operations);
(b) $\approx$ is a congruence with respect to $G^{(n)}$;
(c) the set $\bigcup_n G^{(n)}$ of all gates is closed under composition. In other words: for any $G_1, \ldots, G_n$ defined on $\mathfrak{D}^{(i_1)}, \ldots, \mathfrak{D}^{(i_n)}$, respectively, there is a gate $G_1 \otimes \cdots \otimes G_n$, defined on $\mathfrak{D}^{(i_1+\cdots+i_n)}$, that satisfies the following condition:

$$[G_1 \otimes \cdots \otimes G_n](\rho^{(i_1)} \otimes \cdots \otimes \rho^{(i_n)}) = G_1(\rho^{(i_1)}) \otimes \cdots \otimes G_n(\rho^{(i_n)});$$

(10) $\mathtt{Not}$, $\mathtt{I}$, $\sqrt{\mathtt{I}}$, $\mathtt{T}$ are maps that assume as values gates that belong to $\bigcup_n \mathfrak{G}^{(n)}$.
(11) $\mathtt{Not}$ associates to any $n \geq 1$ the gate $\mathtt{Not}^{(n)}$ (defined on $\mathfrak{D}^{(n)}$) that satisfies the following conditions:
(a) $\mathtt{Not}^{(n)}(|x_1, \ldots, x_n\rangle) \approx |x_1, \ldots, x_{n-1}, 1 - x_n\rangle$;
(b) $\mathtt{Not}^{(n)}(\mathtt{Not}^{(n)}(\rho^{(n)})) \approx \rho^{(n)}$.
(12) $\mathtt{I}$ associates to any $n \geq 1$ the gate $\mathtt{I}^{(n)}$ (defined on $\mathfrak{D}^{(n)}$) that satisfies the following condition: $\mathtt{I}^{(n)}(\rho^{(n)}) = \rho^{(n)}$.
(13) $\sqrt{\mathtt{I}}$ associates to any $n \geq 1$ the gate $\sqrt{\mathtt{I}}^{(n)}$ (defined on $\mathfrak{D}^{(n)}$) that satisfies the following conditions:
(a) $\sqrt{\mathtt{I}}^{(n)}(|x_1, \ldots, x_n\rangle) \approx |x_1, \ldots, x_{n-1}\rangle \otimes \sqrt{\mathtt{I}}^{(1)}(|x_n\rangle)$;
(b) $\sqrt{\mathtt{I}}^{(n)}(\sqrt{\mathtt{I}}^{(n)}(\rho^{(n)})) \approx \rho^{(n)}$;
(c) $\sqrt{\mathtt{I}}^{(n)}(|x_1, \ldots, x_n\rangle)\clubsuit^n|x_1, \ldots, x_n\rangle$
$\sqrt{\mathtt{I}}^{(n)}(|x_1, \ldots, x_n\rangle)\clubsuit^n|x_1, \ldots, 1 - x_n\rangle$.
(14) For any $m, n \geq 1$, $\mathtt{T}$ associates to the triplet $(m, n, 1)$ the gate $\mathtt{T}^{(m,n,1)}$, defined on $\mathfrak{D}^{(m+n+1)}$. We put:

$$\mathtt{And}(\rho^{(m)}, \sigma^{(n)}) := \mathtt{T}^{(m,n,1)}(\rho^{(m)} \otimes \sigma^{(n)} \otimes |0\rangle);$$

$$\mathtt{Or}(\rho^{(m)}, \sigma^{(n)}) := \mathtt{Not}^{(m+n+1)}(\mathtt{And}(\mathtt{Not}^{(m)}(\rho^{(m)}), \mathtt{Not}^{(n)}(\sigma^{(n)}))).$$

The following conditions hold:
(a) $\mathtt{T}^{(m,n,1)}(|x_1, \ldots, x_m, y_1, \ldots, y_n, z\rangle \approx |x_1, \ldots, x_m, y_1, \ldots, y_n, x_m \cdot y_n \widehat{+} z\rangle$, where $\widehat{+}$ is the sum modulo 2;
(b) $\mathtt{And}(\rho, \sigma) \approx \mathtt{And}(\sigma, \rho)$ (commutativity);
(c) $\mathtt{And}(\rho, \mathtt{And}(\sigma, \tau)) \approx \mathtt{And}(\mathtt{And}(\rho, \sigma), \tau)$ (associativity);
(d) $\mathtt{And}(\rho, \mathtt{Or}(\sigma, \tau)) \preceq \mathtt{Or}(\mathtt{And}(\rho, \sigma), \mathtt{And}(\rho, \tau))$ (semidistributivity).
(e) $\sqrt{\mathtt{I}}^{(1)}|0\rangle \preceq \sqrt{\mathtt{I}}^{(m+n+1)}(\mathtt{T}^{(m,n,1)}(\rho^{(m)} \otimes \sigma^{(n)} \otimes |0\rangle)) \preceq \sqrt{\mathtt{I}}^{(1)}|1\rangle$.

In this framework one can naturally define an abstract notion of *entangled quregister*.

Definition 2.10 (Entangled quregister) A quregister $|\psi\rangle \in \mathcal{Q}^{(2n)}$ is called *entangled in the abstract quantum computational structure* $\mathfrak{A}$ iff there are two proper qumixes $\rho^{(n)}$ and $\sigma^{(n)}$ in $\mathfrak{A}$ such that:

- $Red^1(|\psi\rangle) = \rho^{(n)}$ and $Red^2(|\psi\rangle) = \sigma^{(n)}$;
- $\rho^{(n)} \approx \sigma^{(n)}$.

One also says that the qumixes $\rho^{(n)}$ and $\sigma^{(n)}$ are entangled in $\mathfrak{A}$.

One can easily show that the notion of abstract quantum computational structure represents a "good" abstraction from Hilbert-space qumixes.

Consider the structure

$$(\mathfrak{D}, Red, \clubsuit, \spadesuit, \preceq, \mathfrak{G}, \mathtt{Not}, \mathtt{I}, \sqrt{\mathtt{I}}, \mathtt{T}, |0\rangle, |1\rangle),$$

where:

- $\mathfrak{D} = \mathfrak{D}^{\mathcal{H}}$ (the set of all concrete qumixes);
- registers, quregisters and proper qumixes are defined in the expected way;
- *Red* is the standard reduced state function (defined according to the quantum theoretic formalism);
- $\clubsuit^n$ is defined as follows: $\rho^{(n)} \clubsuit^n \sigma^{(n)}$ iff $\rho^{(n)}$ and $\sigma^{(n)}$are two non-orthogonal density operators in the space $\mathcal{H}^{(n)}$ (in other words, for any qumix $\rho \in \mathcal{H}^{(n)}$, $\mathtt{Tr}(\rho(\rho^{(n)} + \sigma^{(n)})) \leq 1$);
- $\spadesuit^n$ is defined as follows: $\rho^{(n)} \spadesuit^n \sigma^{(n)}$ iff

  (1) $\rho^{(n)} = \sum_i c_i P_{|\psi_i\rangle}$ and $\sigma^{(n)} = \sum_i c_i P_{|\varphi_i\rangle}$;
  (2) for at least one $c_i \neq 0$, $|\psi_i\rangle = |\varphi_i\rangle$.

- the relation $\preceq$, the gate-map $\mathfrak{G}$, the gates $\mathtt{Not}^{(n)}$, $\mathtt{I}^{(n)}$, $\sqrt{\mathtt{I}}^{(n)}$, $\mathtt{T}^{(n,m,1)}$ and the two bits $|0\rangle$, $|1\rangle$ are defined in the expected way (according to the definitions given above).

This structure satisfies our definition of abstract quantum computational structure.

3 An Abstract Holistic Semantics

We will now generalize the Hilbert-space quantum computational semantics to the abstract case.[2] Let us first introduce a formal (sentential) language $\mathcal{L}$ for the abstract quantum computational structures. The language contains two privileged atomic sentences **t** and **f**, representing the truth-values *Truth* and *Falsity*, respectively. We will use $\mathbf{q}, \mathbf{r}, \ldots$ as metavariables for atomic sentences, and $\alpha, \beta, \ldots$ as metavariables for sentences. The connectives of $\mathcal{L}$ are: the negation $\neg$, the squareroot of the identity $\sqrt{id}$, a ternary conjunction $\bigwedge$ (which corresponds to the Toffoli gate), the composition-connective $\sharp$ (which corresponds to the abstract tensor product $\otimes$). For any sentences α and β, the expressions $\neg\alpha$, $\sqrt{id}\,\alpha$, $\bigwedge(\alpha, \beta, \mathbf{f})$ (the *ternary conjunction* of α, $\beta, \mathbf{f}$) are sentences. For any sentences $\beta_1, \ldots, \beta_n$, the expression $\sharp(\beta_1, \ldots, \beta_n)$ (the *composition* of $\beta_1, \ldots \beta_n$) is a sentence. The connectives $\neg$, $\sqrt{id}$, $\bigwedge$ are called *gate-connectives*. We will use the following metalinguistic abbreviations:

$$\alpha \wedge \beta := \bigwedge(\alpha, \beta, \mathbf{f}); \; \beta_1 \sharp \cdots \sharp \beta_n := \sharp(\beta_1, \ldots, \beta_n).$$

Before defining the basic notions of the abstract holistic semantics, let us first introduce some useful syntactical notions.

[2]The concrete holistic quantum computational semantics (for a somewhat different language) has been presented in [5].

Definition 3.1

- α is called a gate-sentence iff either α is atomic or the principal connective of α is a gate-connective.
- α is called a compositional sentence iff $\alpha = \beta_1 \sharp \cdots \sharp \beta_m$, where $\beta_1, \ldots, \beta_m$ are gate-sentences.

Definition 3.2 (The atomic complexity of a sentence) The atomic complexity $At(\alpha)$ of a sentence α is the number of occurrences of atomic sentences in α.

For instance, $At(\bigwedge(\mathbf{q}, \mathbf{q}, \mathbf{f})) = 3$. We will also indicate by $\alpha^{(n)}$ a sentence whose atomic complexity is n. The notion of atomic complexity plays an important semantic role. As happens in the case of concrete quantum computational semantics, the meaning of any sentence whose atomic complexity is n is supposed to live in the domain $\mathfrak{D}^{(n)}$. For this reason, $\mathfrak{D}^{(At(\alpha))}$ (briefly indicated by $\mathfrak{D}^{\alpha}$) will be also called the *semantic space* of α.

Any sentence α can be naturally decomposed into its parts, giving rise to a special configuration called the *syntactical tree* of α (indicated by $STree^{\alpha}$).

Roughly, $STree^{\alpha}$ can be represented as a finite sequence of *levels*:

$$\begin{gathered} Level_k(\alpha) \\ \vdots \\ Level_1(\alpha), \end{gathered}$$

where:

- each $Level_i(\alpha)$ (with $1 \leq i \leq k$) is a compositional sentence $\beta_1 \sharp \cdots \sharp \beta_m$ that contains the atomic sentences of α;
- the *bottom level* ($Level_1(\alpha)$) is α;
- the *top level* ($Level_k(\alpha)$) is the sentence $\mathbf{q}_1 \sharp \cdots \sharp \mathbf{q}_t$, where $\mathbf{q}_1, \ldots, \mathbf{q}_t$ are the atomic occurrences in α;
- for any i (with $1 \leq i < k$), $Level_{i+1}(\alpha)$ is the compositional sentence obtained by dropping the *principal gate-connective* in all molecular gate-sentences occurring at $Level_i(\alpha)$, and by repeating all the atomic sentences that possibly occur at $Level_i(\alpha)$.

By *Height* of α (indicated by *Height*(α)) we mean the number of levels of the syntactical tree of α.

More precisely, the syntactical tree of a sentence (whose atomic complexity is t) is defined as follows.

Definition 3.3 (The syntactical tree of α) The syntactical tree of α is the following sequence of sentences:

$$STree^{\alpha} = (Level_1(\alpha), \ldots, Level_k(\alpha)),$$

where:

- $Level_1(\alpha) = \alpha$;
- $Level_{i+1}$ is defined as follows for any i such that $1 \le i < k$. The following cases are possible:
  (1) $Level_i(\alpha)$ does not contain any gate-connective. Hence, $Level_i(\alpha) = \mathbf{q}_1 \sharp \cdots \sharp \mathbf{q}_t$ and $Height(\alpha) = i$;
  (2) $Level_i(\alpha)$ is the compositional sentence $\beta_1 \sharp \cdots \sharp \beta_m$, and for at least one j, the principal connective of β_j is a gate-connective. Consider the following sequence of sentences:

$$\beta'_1, \ldots, \beta'_m,$$

$$\text{where: } \beta'_h = \begin{cases} \beta_h, & \text{if } \beta_h \text{ is atomic;} \\ \beta^*_h, & \text{otherwise.} \end{cases}$$

$$\text{Where: } \beta^*_h = \begin{cases} \delta, & \text{if } \beta_h = \neg\delta \text{ or } \beta_h = \sqrt{id}\,\delta; \\ \gamma \sharp \delta \sharp \mathbf{f}, & \text{if } \beta_h = \bigwedge(\gamma, \delta, \mathbf{f}). \end{cases}$$

Then,

$$Level_{i+1}(\alpha) = \beta'_1 \sharp \cdots \sharp \beta'_m.$$

As an example, consider the following sentence: $\alpha = \mathbf{q} \wedge \neg\mathbf{q} = \bigwedge(\mathbf{q}, \neg\mathbf{q}, \mathbf{f})$. The syntactical tree of α is the following sequence of levels:

$$Level_3(\alpha) = \mathbf{q} \sharp \mathbf{q} \sharp \mathbf{f};$$

$$Level_2(\alpha) = \mathbf{q} \sharp \neg\mathbf{q} \sharp \mathbf{f};$$

$$Level_1(\alpha) = \bigwedge(\mathbf{q}, \neg\mathbf{q}, \mathbf{f}).$$

Clearly, $Height(\bigwedge(\mathbf{q}, \neg\mathbf{q}, \mathbf{f})) = 3$.

The syntactical tree of α (which represents a purely syntactical object) uniquely determines a sequence of abstract gates that are all defined on the semantic space of α. This gate-sequence is called the *qumix tree* of α. Let α be a sentence such that $Height(\alpha) = k$.

Definition 3.4 (The qumix tree of α) The qumix tree of α is the sequence of abstract gates

$$QumTree^{\alpha} = (G^{\alpha}_{k-1}, \ldots, G^{\alpha}_1),$$

that is defined as follows. Suppose that

$$Level_{i-1}(\alpha) = \beta_1^{(t_1)} \sharp \cdots \sharp \beta_m^{(t_m)},$$

(where $1 \le i \le k$). We put:

$$G^{\alpha}_{i-1} = X_1^{(t_1)} \otimes \cdots \otimes X_m^{(t_m)},$$

where any $X_j^{(t_j)}$ is a gate defined on $\mathfrak{D}^{(t_j)}$ such that:

$$X_j^{(t_j)} = \begin{cases} \mathtt{I}^{(t_j)}, & \text{if } \beta_j^{(t_j)} \text{ is atomic;} \\ \mathtt{Not}^{(t_j)}, & \text{if } \beta_j^{(t_j)} = \neg\delta; \\ \sqrt{\mathtt{I}}^{(t_j)}, & \text{if } \beta_j^{(t_j)} = \sqrt{id}\delta; \\ \mathtt{T}^{(r,s,1)}, & \text{if } \beta_j^{(t_j)} = \bigwedge(\gamma^{(r)}, \delta^{(s)}, \mathbf{f}). \end{cases}$$

Consider now a sentence α and let $(G_{k-1}^{\alpha}, \dots, G_1^{\alpha})$ be the qumix tree of α. Any choice of a qumix ρ in $\mathfrak{D}^{\alpha}$ determines a sequence $(\rho_k, \dots, \rho_1)$ of qumixes of $\mathfrak{D}^{\alpha}$, where:

$$\begin{aligned} \rho_k &= \rho \\ \rho_{k-1} &= G_{k-1}^{\alpha}(\rho_k) \\ &\vdots \\ \rho_1 &= G_1^{\alpha}(\rho_2). \end{aligned}$$

The qumix ρ_k can be regarded as a possible *input-information* concerning the atomic parts of α, while ρ_1 represents the *output-information* about α, given the input-information ρ_k. Each ρ_i corresponds to the *information* about $Level_i(\alpha)$, given the input-information ρ_k.

How to determine an information about the parts of α under a given input? It is natural to apply our abstract reduced information map.

Consider the syntactical tree of α and suppose that:

$$Level_i(\alpha) = \beta_{i_1} \sharp \cdots \sharp \beta_{i_r}.$$

We know that the qumix tree of α and the choice of an input ρ_k (in $\mathfrak{D}^{\alpha}$) determine a sequence of qumixes:

$$\begin{gathered} \rho_k \leftrightsquigarrow Level_k(\alpha) = \mathbf{q}_1 \sharp \cdots \sharp \mathbf{q}_t \\ \vdots \\ \rho_i \leftrightsquigarrow Level_i(\alpha) = \beta_{i_1} \sharp \cdots \sharp \beta_{i_r} \\ \vdots \\ \rho_1 \leftrightsquigarrow Level_1(\alpha) = \alpha \end{gathered}$$

We can consider $Red^j(\rho_i)$, the *reduced information of* ρ_i *with respect to the* j*-th part*. From a semantic point of view, this object can be regarded as a *contextual information* about β_{i_j} (the subformula of α occurring at the j-th position at $Level_i(\alpha)$) under the input ρ_k.

We will now give the basic definitions of the abstract semantics. The main concept is the notion of *(abstract) quantum computational model*: a map that assigns to any sentence α of the language $\mathcal{L}$ a *meaning*, a qumix living in the semantic space $\mathfrak{D}^{\alpha}$. Of course (like in the standard semantic approaches), the map shall respect the logical form of α. In the compositional semantics, the meaning of any sentence is determined by the meanings of its parts (from the parts to the whole). In the holistic semantics, instead, a model assigns to any sentence a *global meaning* that determines the contextual meanings of all its parts (from the whole to the parts). It may happen that one and the same sentence receives different meanings in different contexts.

Let us first introduce the notion of *(abstract) compositional quantum computational model* (briefly, *compositional model*).

Definition 3.5 (Compositional model) A *compositional model* is a map $\mathtt{Qum}$ that associates to any sentence α of $\mathcal{L}$ a qumix in the semantic space $\mathfrak{D}^{\alpha}$. The following conditions hold:

(1) $\mathtt{Qum}(\mathbf{f}) = |0\rangle$; $\mathtt{Qum}(\mathbf{t}) = |1\rangle$;
(2) $\mathtt{Qum}(\neg\beta) = \mathtt{Not}^{(At(\beta))}(\beta)$;
(3) $\mathtt{Qum}(\sqrt{id}\,\beta) = \sqrt{\mathtt{I}}^{(At(\beta))}(\beta)$;
(4) $\mathtt{Qum}(\bigwedge(\beta,\gamma,\mathbf{f})) = \mathtt{T}^{(At(\beta),At(\gamma),At(\mathbf{f}))}(\mathtt{Qum}(\beta)\otimes\mathtt{Qum}(\gamma)\otimes\mathtt{Qum}(\mathbf{f}))$;
(5) $\mathtt{Qum}(\beta_1\sharp\cdots\sharp\beta_n) = \mathtt{Qum}(\beta_1)\otimes\cdots\otimes\mathtt{Qum}(\beta_n)$.

Compositional models are clearly context-independent. Any $\mathtt{Qum}$ determines a meaning for each level of the syntactical tree of any sentence α. Suppose that $Level_{Height(\alpha)}(\alpha) = \mathbf{q}_1\sharp\cdots\sharp\mathbf{q}_t$ and let $G^{\alpha}_{Height(\alpha)-1},\ldots,G^{\alpha}_1$ be the qumix tree of α. We put:
$\mathtt{Qum}(Level_{Height(\alpha)}(\alpha)) = \mathtt{Qum}(\mathbf{q}_1)\otimes\cdots\otimes\mathtt{Qum}(\mathbf{q}_t)$;
$\mathtt{Qum}(Level_i(\alpha)) = G^{\alpha}_i(\mathtt{Qum}(Level_{i+1}(\alpha)))$, for any $i < Height(\alpha)$.

The notion of logical consequence in the framework of the compositional semantics is defined in the expected way.

Definition 3.6 (Consequence in a compositional model $\mathtt{Qum}$) A sentence β is a consequence of a sentence α in a compositional model $\mathtt{Qum}$ ($\alpha \models_{\mathtt{Qum}} \beta$) iff $\mathtt{Qum}(\alpha) \preceq \mathtt{Qum}(\beta)$, where $\preceq$ is the preorder relation defined on $\mathfrak{D}$.

Definition 3.7 (Logical consequence (in the compositional semantics)) A sentence β is a consequence of a sentence α (in the compositional semantics) iff for any $\mathtt{Qum}$,

$$\alpha \models_{\mathtt{Qum}} \beta.$$

We call *abstract compositional quantum computational logic* (abbreviated as **AbCQCL**) the logic that is semantically characterized by the logical consequence relation we have just defined. (Hence, $\alpha \models_{\mathbf{AbCQCL}} \beta$ iff for any $\mathtt{Qum}$, $\alpha \models_{\mathtt{Qum}} \beta$.)

Let us now turn to the holistic semantics. We will distinguish three possible "levels of semantic holism", which correspond to three different notions of holistic model. We will speak of *superholistic semantics*, *normal holistic semantics* and *locally compositional semantics*, respectively.

The superholistic semantics is the most liberal one: sentences may receive different meanings even in the framework of one and the same context. In other words, different occurrences of one and the same subformula in a given sentence may have different contextual meanings. Such a liberal point of view might appear somewhat strange in the case of scientific languages. However, it is quite reasonable for natural and artistic languages. Consider a very "long" expression γ (for instance, a novel or a musical score): why should all occurrences of a part of γ have a constant contextual meaning?[3]

The normal holistic semantics is more restrictive: although sentences may receive different meanings in different contexts, all occurrences of a subformula in a given sentence receive a constant contextual meaning. Finally, the locally compositional semantics assumes a further restriction: the map that assigns contextual meanings to all subformulas of a given formula should always be simulated by a compositional model. In other words, once fixed a given context, contextual meanings are supposed to behave in a compositional way.

We will give now the technical definitions of the notions of *superholistic model*, *normal holistic model* and *locally compositional model*.

Definition 3.8 (Superholistic model) A *superholistic model* of the language $\mathcal{L}$ is a map $\mathtt{Hol}$ that associates a meaning $\mathtt{Hol}(Level_i(\alpha))$ to each level of the syntactical tree of α, for any sentence α of $\mathcal{L}$. The following conditions are required:

(1) $\mathtt{Hol}(Level_i(\alpha)) \in \mathfrak{D}^\alpha$.
In other words, the meaning of $Level_i(\alpha)$ under $\mathtt{Hol}$ belongs to the semantic space of α.

(2) Let $(G^\alpha_{Height(\alpha)-1}, \ldots, G^\alpha_1)$ be the qumix tree of α and let $1 \le i < Height(\alpha)$. Then,

$$\mathtt{Hol}(Level_i(\alpha)) = G^\alpha_i(\mathtt{Hol}(Level_{i+1}(\alpha))).$$

In other words the global meaning of each level (different from the top level) is obtained by applying the corresponding gate to the meaning of the level that occurs immediately above.

(3) Let $Level_i(\alpha) = \beta_1 \sharp \ldots \sharp \beta_r$. Then:
$\beta_j = \mathbf{f} \curvearrowright Red^j(\mathtt{Hol}(Level_i(\alpha))) = |0\rangle$;
$\beta_j = \mathbf{t} \curvearrowright Red^j(\mathtt{Hol}(Level_i(\alpha))) = |1\rangle$, for any j $(1 \le j \le r)$.
In other words, the contextual meanings of $\mathbf{f}$ and of $\mathbf{t}$ are always the *Falsity* and the *Truth*, respectively.

On this basis, we put:

$$\mathtt{Hol}(\alpha) := \mathtt{Hol}(Level_1(\alpha)),$$

for any sentence α.

Unlike *compositional* semantics, any $\mathtt{Hol}(\alpha)$ represents a kind of autonomous semantic context that is not necessarily correlated with the meanings of other sentences.

[3] As observed by the conductor Piero Bellugi, the "copy and paste-operation" cannot exist in music! An application of a quantum-like holistic semantics to a formal analysis of musical languages has been studied in [6].

At the same time, given a sentence γ, $\mathtt{Hol}$ determines the *contextual meaning*, with respect to the context $\mathtt{Hol}(\gamma)$, of any *occurrence of a subformula* β in γ.

Definition 3.9 (Contextual meaning of a subformula) Let β be a subformula of γ.

1. Suppose that β is a gate-sentence and let $\beta[^i_j]$ be an occurrence of β at the j^{th}-position of the i^{th}-level of the syntactical tree of γ. Then

$$\mathtt{Hol}^\gamma(\beta[^i_j]) := \mathit{Red}^j(\mathtt{Hol}(\mathit{Level}_i(\gamma))).$$

2. Suppose that β is a compositional sentence such that

$$\beta[^i_j] = \beta_1[^i_{j_1}]\sharp\cdots\sharp\beta_m[^i_{j_m}].$$

Then,

$$\mathtt{Hol}^\gamma(\beta[^i_j]) = \begin{cases} \mathtt{Hol}(\mathit{Level}_i(\gamma)), & \text{if } \mathit{Level}_i(\gamma) = \beta_1\sharp\cdots\sharp\beta_m; \\ \mathtt{Hol}^\gamma(\beta_1[^i_{j_1}]) \otimes\cdots\otimes \mathtt{Hol}^\gamma(\beta_m[^i_{j_m}]), & \text{otherwise.} \end{cases}$$

(Notice that $\mathtt{Hol}^\gamma(\beta_1[^i_{j_1}]\sharp\cdots\sharp\beta_m[^i_{j_m}])$ is well defined. For, by definition of syntactical tree, two different levels of STree^γ cannot have the same form $\beta_1\sharp\cdots\sharp\beta_m$.)

Hence, in particular, we have for any sentence γ:

$$\mathtt{Hol}^\gamma(\gamma) = \mathtt{Hol}(\mathit{Level}_1(\gamma)) = \mathtt{Hol}(\gamma).$$

Apparently, $\mathtt{Hol}^\gamma$ is a partial function that only assigns meanings to the occurrences of subformulas of γ. Given a formula γ, we will call the partial function $\mathtt{Hol}^\gamma$ a *contextual holistic model* of the language.

Definition 3.10 (Normal holistic model) A *normal holistic model* of the language $\mathcal{L}$ is a superholistic model $\mathtt{Hol}$ that satisfies the following condition: if $\beta[^i_j]$ and $\beta[^h_k]$ are two nodes of the syntactical tree of γ, representing two occurrences of the same gate-subformula β, then

$$\mathtt{Hol}^\gamma(\beta[^i_j]) = \mathtt{Hol}^\gamma(\beta[^h_k]).$$

As a consequence, two different occurrences of one and the same subformula in a sentence γ receive the same contextual meaning with respect to the context $\mathtt{Hol}(\gamma)$.[4]

For normal holistic models, one can naturally define the *contextual meaning* of *any subformula* β of γ, with respect to the context $\mathtt{Hol}(\gamma)$.

Definition 3.11 Let β be a subformula of γ.

[4]In the definition of holistic model given in [5] we have only required that two different occurrences of one and the same atomic subformula at the top level of the syntactical tree of a sentence γ receive the same contextual meaning with respect to the context $\mathtt{Hol}(\gamma)$. However this is not sufficient to prove that all occurrences of one and the same subformula of a given sentence receive the same contextual meaning.

1. Suppose that β is a gate-sentence. Then

$$\mathtt{Hol}^{\gamma}(\beta) := \mathtt{Hol}^{\gamma}(\beta[^i_j]),$$

where $\beta[^i_j]$ is any occurrence of β as a node of $STree^{\gamma}$.
2. Suppose that β is a compositional sentence $\beta_1 \sharp \cdots \sharp \beta_m$ (where $\beta_1, \ldots, \beta_m$ are gate-sentences). Then,

$$\mathtt{Hol}^{\gamma}(\beta) = \begin{cases} \mathtt{Hol}(Level_i(\gamma)), & \text{if } Level_i(\gamma) = \beta_1 \sharp \cdots \sharp \beta_m; \\ \mathtt{Hol}^{\gamma}(\beta_1) \otimes \cdots \otimes \mathtt{Hol}^{\gamma}(\beta_m), & \text{otherwise.} \end{cases}$$

Suppose now that β is a subformula of two different formulas γ and δ. Generally, we have:

$$\mathtt{Hol}^{\gamma}(\beta) \neq \mathtt{Hol}^{\delta}(\beta).$$

In other words, sentences may receive different contextual meanings in different contexts also in the case of the normal holistic semantics.

Definition 3.12 (Locally compositional model) A locally compositional model of the language $\mathcal{L}$ is a normal holistic model $\mathtt{Hol}$ that satisfies the following condition: for any γ there exists a compositional model $\mathtt{Qum}$ such that for any subformula β of γ, $\mathtt{Hol}^{\gamma}(\beta) \approx \mathtt{Qum}(\beta)$.

One can easily realize that compositional models are special cases of holistic models.

Lemma 3.1 *Any compositional model* $\mathtt{Qum}$ *uniquely determines a locally compositional model* $\mathtt{Hol}$ *such that*:

1. $\mathtt{Hol}(\alpha) = \mathtt{Qum}(\alpha)$, *for any sentence* α;
2. $\mathtt{Hol}^{\gamma}(\alpha) = \mathtt{Qum}(\alpha)$, *for any* α *and for any* γ *such that* α *is a subformula of* γ.

Proof Given $\mathtt{Qum}$ we can define $\mathtt{Hol}$ as follows:

$$\mathtt{Hol}(Level_{Height(\alpha)}(\alpha)) = \mathtt{Qum}(\mathbf{q}_1) \otimes \cdots \otimes \mathtt{Qum}(\mathbf{q}_t),$$

for any α such that $Level_{Height(\alpha)}(\alpha) = \mathbf{q}_1 \sharp \cdots \sharp \mathbf{q}_t$. □

The notion of logical consequence in the framework of the abstract holistic semantics represents a reasonable variant of the standard notions of logical consequence. As expected, the three different concepts of holistic model give rise to three different concepts of logical consequence:

a) $\alpha \vDash_S \beta$ (β is a logical consequence of α in the *superholistic semantics*);
b) $\alpha \vDash_N \beta$ (β is a logical consequence of α in the *normal holistic semantics*);
c) $\alpha \vDash_L \beta$ (β is a logical consequence of α in the *locally compositional semantics*);

Let us first define the notion of *consequence in a given contextual model*.

Definition 3.13 (Consequence in a given contextual model $\mathtt{Hol}^\gamma$) Let γ be a sentence and let $\mathtt{Hol}$ be a model, which may be either superholistic or normal holistic or locally compositional. A sentence β is a consequence of a sentence α in the contextual model $\mathtt{Hol}^\gamma$ ($\alpha \models_{\mathtt{Hol}^\gamma} \beta$) iff

1. α and β are subformulas of γ;
2. $\mathtt{Hol}^\gamma(\alpha_i) \preceq \mathtt{Hol}^\gamma(\beta_j)$, for at least one occurrence α_i of α in γ and for at least one occurrence β_j of β in γ.

Definition 3.14 (Logical consequence (in the holistic semantics))

a) $\alpha \vDash_S \beta$ (β is a logical consequence of α in the superholistic semantics) iff for any sentence γ such that α and β are subformulas of γ and for any superholistic model $\mathtt{Hol}$,

$$\alpha \models_{\mathtt{Hol}^\gamma} \beta.$$

b) $\alpha \vDash_N \beta$ (β is a logical consequence of α in the normal holistic semantics) iff for any sentence γ such that α and β are subformulas of γ and for any normal holistic model $\mathtt{Hol}$,

$$\alpha \models_{\mathtt{Hol}^\gamma} \beta.$$

c) $\alpha \vDash_L \beta$ (β is a logical consequence of α in the locally compositional semantics) iff for any sentence γ such that α and β are subformulas of γ and for any locally compositional model $\mathtt{Hol}$,

$$\alpha \models_{\mathtt{Hol}^\gamma} \beta.$$

On this basis, we obtain three different forms of abstract quantum computational logics that are characterized, respectively, by the three logical consequence relations we have just defined. We will call these logics: *abstract superholistic quantum computational logic* (abbreviated as $\mathbf{AbH^SQCL}$), *abstract normal holistic quantum computational logic* ($\mathbf{AbH^NQCL}$) and *abstract locally compositional quantum computational logic* ($\mathbf{AbH^LQCL}$).

We want now study the relationships that hold between these different forms of holistic logic. First of all we show that the locally compositional semantics and the compositional semantics characterize the same logic.

Theorem 3.1 *For any sentences* α *and* β,

$$\alpha \models_{\mathbf{AbH^LQCL}} \beta \quad \textit{iff} \quad \alpha \models_{\mathbf{AbCQCL}} \beta.$$

Proof Let us write $\alpha \models_{\mathbf{C}} \beta$ and $\alpha \models_{\mathbf{H^L}} \beta$ instead of $\alpha \models_{\mathbf{AbCQCL}} \beta$ and $\alpha \models_{\mathbf{AbH^LQCL}} \beta$, respectively. We prove:

1. $\alpha \models_{\mathbf{H^L}} \beta \curvearrowright \alpha \models_{\mathbf{C}} \beta$.
2. $\alpha \models_{\mathbf{C}} \beta \curvearrowright \alpha \models_{\mathbf{H^L}} \beta$.

Proof of 1.
Suppose that $\alpha \models_{\mathbf{H^L}} \beta$ and $\alpha \not\models_{\mathbf{C}} \beta$. Hence, there exists a compositional model $\mathtt{Qum}$ such that $\mathtt{Qum}(\alpha) \not\preceq \mathtt{Qum}(\beta)$. Take $\alpha \wedge \beta$. By Lemma 3.1, $\mathtt{Qum}$ determines a locally compositional model $\mathtt{Hol}$ such that:

$$\mathtt{Hol}^{\alpha\wedge\beta}(\alpha) = \mathtt{Qum}(\alpha); \qquad \mathtt{Hol}^{\alpha\wedge\beta}(\beta) = \mathtt{Qum}(\beta).$$

Hence, $\mathtt{Hol}^{\alpha\wedge\beta}(\alpha) \not\preceq \mathtt{Hol}^{\alpha\wedge\beta}(\beta)$, against the hypothesis.
Proof of 2.
Suppose that $\alpha \models_{\mathbf{C}} \beta$ and $\alpha \not\models_{\mathbf{H^L}} \beta$. Then, there exists a holistic model $\mathtt{Hol}$ and a sentence γ such that:

- α and β are subformulas of γ;
- $\mathtt{Hol}^{\gamma}(\alpha) \not\preceq \mathtt{Hol}^{\gamma}(\beta)$.

By definition of locally compositional model there exists a compositional model $\mathtt{Qum}$ such that $\mathtt{Qum}(\alpha) \not\preceq \mathtt{Qum}(\beta)$, against the hypothesis. □

We will now prove that **AbH$^{\mathbf{N}}$QCL** is strictly weaker than **AbH$^{\mathbf{L}}$QCL**, while **AbH$^{\mathbf{S}}$QCL** is strictly weaker than **AbH$^{\mathbf{N}}$QCL**.

Theorem 3.2 *$\alpha \vDash_{\mathbf{AbH^NQCL}} \beta \curvearrowright \alpha \vDash_{\mathbf{AbH^LQCL}} \beta$, but not the other way around.*

Proof

1. $\alpha \vDash_{\mathbf{AbH^NQCL}} \beta \curvearrowright \alpha \vDash_{\mathbf{AbH^LQCL}} \beta$, because locally compositional models are special examples of normal holistic models.
2. $\alpha \vDash_{\mathbf{AbH^LQCL}} \beta \not\curvearrowright \alpha \vDash_{\mathbf{AbH^NQCL}} \beta$.
   Consider the following counterexample in the Hilbert-space semantics. Let $\alpha = \bigwedge(\mathbf{q}, \neg\mathbf{q}, \mathbf{f})$, $\beta = \bigwedge(\sqrt{id}\,\mathbf{f}, \sqrt{id}\,\mathbf{f}, \mathbf{f})$, $\gamma = \bigwedge(\alpha, \beta, \mathbf{f})$. We define a normal holistic $\mathtt{Hol}$ that assigns to the top level of the syntactical tree of γ a pure state, whose first component is entangled. The syntactical tree of γ is:

$$Level_4(\gamma) = \mathbf{q} \sharp \mathbf{q} \sharp \mathbf{f} \sharp \mathbf{f} \sharp \mathbf{f} \sharp \mathbf{f} \sharp \mathbf{f};$$

$$Level_3(\gamma) = \mathbf{q} \sharp \neg\mathbf{q} \sharp \mathbf{f} \sharp \sqrt{id}\,\mathbf{f} \sharp \sqrt{id}\,\mathbf{f} \sharp \mathbf{f} \sharp \mathbf{f};$$

$$Level_2(\gamma) = \bigwedge(\mathbf{q}, \neg\mathbf{q}, \mathbf{f}) \sharp \bigwedge\left(\sqrt{id}\,\mathbf{f}, \sqrt{id}\,\mathbf{f}, \mathbf{f}\right) \sharp \mathbf{f};$$

$$Level_1(\gamma) = \bigwedge\left(\bigwedge(\mathbf{q}, \neg\mathbf{q}, \mathbf{f}), \bigwedge\left(\sqrt{id}\,\mathbf{f}, \sqrt{id}\,\mathbf{f}, \mathbf{f}\right), \mathbf{f}\right).$$

   Hence, the qumix tree of γ is:

$$\mathcal{D}_{\mathtt{I}^{(1)}} \otimes \mathcal{D}_{\mathtt{Not}^{(1)}} \otimes \mathcal{D}_{\mathtt{I}^{(1)}} \otimes \mathcal{D}_{\sqrt{\mathtt{I}}^{(1)}} \otimes \mathcal{D}_{\sqrt{\mathtt{I}}^{(1)}} \otimes \mathcal{D}_{\mathtt{I}^{(1)}} \otimes \mathcal{D}_{\mathtt{I}^{(1)}}$$
$$\mathcal{D}_{\mathtt{T}^{(1,1,1)}} \otimes \mathcal{D}_{\mathtt{T}^{(1,1,1)}} \otimes \mathcal{D}_{\mathtt{I}^{(1)}}$$
$$\mathcal{D}_{\mathtt{T}^{(3,3,1)}}.$$

   Define $\mathtt{Hol}$ as follows:

$\mathtt{Hol}(Level_4(\gamma)) = P_{\frac{1}{\sqrt{2}}(|010\rangle+|100\rangle)} \otimes P_0^{(1)} \otimes P_0^{(1)} \otimes P_0^{(1)} \otimes P_0^{(1)}$.

By applying the gates of the qumix tree of γ to $\mathtt{Hol}(Level_4(\gamma))$ we obtain:

$\mathtt{Hol}(Level_3(\gamma)) = P_{\frac{1}{\sqrt{2}}(|000\rangle+|110\rangle)} \otimes {}^{\mathcal{D}}\sqrt{\mathtt{I}}^{(1)} P_0^{(1)} \otimes {}^{\mathcal{D}}\sqrt{\mathtt{I}}^{(1)} P_0^{(1)} \otimes P_0^{(1)} \otimes P_0^{(1)}$;

$\mathtt{Hol}(Level_2(\gamma)) = P_{\frac{1}{\sqrt{2}}(|000\rangle+|111\rangle)} \otimes {}^{\mathcal{D}}\mathtt{T}^{(1,1,1)}({}^{\mathcal{D}}\sqrt{\mathtt{I}}^{(1)} P_0^{(1)} \otimes {}^{\mathcal{D}}\sqrt{\mathtt{I}}^{(1)} P_0^{(1)} \otimes P_0^{(1)}) \otimes P_0^{(1)}$;

$\mathtt{Hol}(Level_1(\gamma)) = {}^{\mathcal{D}}\mathtt{T}^{(3,3,1)}(P_{\frac{1}{\sqrt{2}}(|000\rangle+|111\rangle)} \otimes {}^{\mathcal{D}}\mathtt{T}^{(1,1,1)}({}^{\mathcal{D}}\sqrt{\mathtt{I}}^{(1)} P_0^{(1)} \otimes {}^{\mathcal{D}}\sqrt{\mathtt{I}}^{(1)} P_0^{(1)} \otimes P_0^{(1)}) \otimes P_0^{(1)})$.

Hence:

$\mathtt{Hol}^{\gamma}(\alpha) = P_{\frac{1}{\sqrt{2}}(|000\rangle+|111\rangle)}$ and $\mathtt{p}(\mathtt{Hol}^{\gamma}(\alpha)) = \frac{1}{2}$;

$\mathtt{Hol}^{\gamma}(\beta) = {}^{\mathcal{D}}\mathtt{T}^{(1,1,1)}({}^{\mathcal{D}}\sqrt{\mathtt{I}}^{(1)} P_0^{(1)} \otimes {}^{\mathcal{D}}\sqrt{\mathtt{I}}^{(1)} P_0^{(1)} \otimes P_0^{(1)})$ and $\mathtt{p}(\mathtt{Hol}^{\gamma}(\beta)) = \frac{1}{4}$.

Consequently: $\alpha \nvDash_{\mathtt{Hol}^{\gamma}} \beta$.

At the same time, one can easily show that for any $\mathtt{Qum}$, $\alpha \models_{\mathtt{Qum}} \beta$. □

The counterexample considered in the proof of Theorem 3.2 clearly shows how entanglement is responsible for the creation of somewhat "pathological" holistic models in comparison with the compositional semantics.[5]

Theorem 3.3 $\alpha \vDash_{\mathbf{AbH^SQCL}} \beta \curvearrowright \alpha \vDash_{\mathbf{AbH^NQCL}} \beta$, *but not the other way around.*

Proof

1. $\alpha \vDash_{\mathbf{AbH^SQCL}} \beta \curvearrowright \alpha \vDash_{\mathbf{AbH^NQCL}} \beta$, because normal holistic models are special examples of superholistic models.
2. $\alpha \vDash_{\mathbf{AbH^NQCL}} \beta \not\curvearrowright \alpha \vDash_{\mathbf{AbH^SQCL}} \beta$. A counterexample is the commutativity-property for the connective conjunction. Consider the following sentence:

$$\gamma = \bigwedge\Big(\bigwedge(\mathbf{q},\mathbf{r},\mathbf{f}), \bigwedge(\mathbf{r},\mathbf{q},\mathbf{f}), \mathbf{f}\Big),$$

whose syntactical tree is:

$$Level_3(\gamma) = \mathbf{q}\sharp\mathbf{r}\sharp\mathbf{f}\sharp\mathbf{r}\sharp\mathbf{q}\sharp\mathbf{f},$$

$$Level_2(\gamma) = \bigwedge(\mathbf{q},\mathbf{r},\mathbf{f})\sharp\bigwedge(\mathbf{r},\mathbf{q},\mathbf{f})\sharp\mathbf{f},$$

$$Level_1(\gamma) = \bigwedge\Big(\bigwedge(\mathbf{q},\mathbf{r},\mathbf{f}), \bigwedge(\mathbf{r},\mathbf{q},\mathbf{f}), \mathbf{f}\Big).$$

[5] In [5] we have wrongly asserted that for any $\mathtt{Hol}$ that satisfies the definition of normal holistic model and for any γ, the partial map $\mathtt{Hol}^{\gamma}$ can always be simulated by a compositional model $\mathtt{Qum}$.

Let $\mathtt{Hol}$ be a superholistic model that is defined as follows for the syntactical tree of γ:

$$\mathtt{Hol}(\mathit{Level}_3(\gamma)) = P_1^{(1)} \otimes P_1^{(1)} \otimes P_0^{(1)} \otimes P_0^{(1)} \otimes P_0^{(1)} \otimes P_0^{(1)}.$$

We obtain:
$\mathtt{Hol}^\gamma(\mathbf{q}[^3_1]) = P_1^{(1)}$; $\mathtt{Hol}^\gamma(\mathbf{r}[^3_2]) = P_1^{(1)}$; $\mathtt{Hol}^\gamma(\mathbf{r}[^3_4]) = P_0^{(1)}$; $\mathtt{Hol}^\gamma(\mathbf{q}[^3_5]) = P_0^{(1)}$. Hence:

$$\mathtt{Hol}^\gamma\Big(\bigwedge(\mathbf{q},\mathbf{r},\mathbf{f})[^i_j]\Big) = P_1^{(1)} \otimes P_1^{(1)} \otimes P_1^{(1)};$$

$$\mathtt{Hol}^\gamma\Big(\bigwedge(\mathbf{r},\mathbf{q},\mathbf{f})[^h_k]\Big) = P_0^{(1)} \otimes P_0^{(1)} \otimes P_0^{(1)},$$

for any occurrence $\bigwedge(\mathbf{q},\mathbf{r},\mathbf{f})[^i_j]$ of $\bigwedge(\mathbf{q},\mathbf{r},\mathbf{f})$ and for any occurrence $\bigwedge(\mathbf{r},\mathbf{q},\mathbf{f})[^h_k]$ of $\bigwedge(\mathbf{r},\mathbf{q},\mathbf{f})$ in the syntactical tree of γ. Consequently: $\bigwedge(\mathbf{q},\mathbf{r},\mathbf{f}) \nvDash_{\mathtt{Hol}^\gamma} \bigwedge(\mathbf{r},\mathbf{q},\mathbf{f})$.

At the same time we have: $\bigwedge(\alpha,\beta,\mathbf{f}) \vDash_{\mathbf{AbH^NQCL}} \bigwedge(\beta,\alpha,\mathbf{f})$, because the Toffoli gate satisfies commutativity in any abstract quantum computational structure. □

Accordingly, $\mathbf{AbH^SQCL}$ seems to be an interesting logical framework that permits us to model the behavior of non-commutative conjunctions. It is worthwhile noticing that, in spite of the "quasi-dialectical" character of $\mathbf{AbH^SQCL}$, the identity principle ($\alpha \vDash_{\mathbf{AbH^SQCL}} \alpha$) remains valid.

As expected, each abstract holistic logic has a natural concrete counterpart, represented by the logic that is characterized by the class of all models based on Hilbert-space quantum computational structures. Let us indicate by $\mathbf{H^XQCL}$ the concrete counterpart of the logic $\mathbf{AbH^XQCL}$, where $\mathbf{X}$ is either $\mathbf{S}$ or $\mathbf{N}$ or $\mathbf{L}$. One can easily show that each abstract holistic logic is weaker than its concrete counterpart.

Theorem 3.4 $\alpha \vDash_{\mathbf{AbH^XQCL}} \beta \curvearrowright \alpha \vDash_{\mathbf{H^XQCL}} \beta$.

Proof Any concrete quantum computational structure is a special example of an abstract quantum computational structure. □

Unlike concrete quantum computational logics, abstract quantum computational logics can be naturally axiomatized. The calculi and the completeness theorems for such logics will be presented elsewhere.

An open problem is the following: are there pairs of sentences α and β such that $\alpha \vDash_{\mathbf{H^XQCL}} \beta$ and $\alpha \nvDash_{\mathbf{AbH^XQCL}} \beta$?

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