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Continuous functions as quantum operations: a probabilistic approximation

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ABSTRACT. In this note we propose a version of the classical Stone-Weierstrass theorem in the context of quantum operations, by introducing a particular class of quantum operations, dubbed *polynomial quantum operations*. This result permits to interpret from a probabilistic point of view, and up to a certain approximation, any continuous function from the real cube $[0,1]^n$ to the real interval $[0,1]$ as a quantum operation.

KEYWORDS: Quantum operations, PMV-algebras, quantum computation.

1. Introduction

Since the classical work of Birkhoff and von Neumann [4], logical and algebraic perspectives of several aspects of quantum theory have been proposed. Leading examples are orthomodular lattices [21], and effect algebras, appeared independently under several names (e.g. D-posets [23]) as a generalization of orthomodular posets [13, 16]. Moreover, effect algebras play a fundamental role

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in various studies on fuzzy probability theory [3, 17] also.
In recent times, quantum computation itself stirred increasing attention, and an array of related algebraic structures arose [5, 6, 18, 15]. Those structures stem from an abstract description of circuits obtained by combinations of quantum gates [7]. Let us mention as examples *quantum MV-algebras* [12], *quasi MV-algebras*, $\sqrt{'}$*quasi MV-algebras* and *product MV-algebras* [25, 14, 10, 28].
Even if those structures are plainly related to quantum computing, some of the functions in their types are algebraic abstractions of irreversible transformations, e.g. the truncated disjunction "$\oplus$" [7].
These observations provides a general framework for a probabilistic-type representation of continuous functions in the real interval $[0,1]$ as *quantum operations* in the sense of [24]. Therefore, in the present paper we show that all those algebraic structures mentioned so far are fully settled into the general model of quantum computing, based on quantum operations acting on density operators.
The irreversible quantum operational approach has plenty of advantages in the implementation of quantum computational devices: as Aharonov, Kitaev and Nisan discussed [1], there are several relevant problems to deal with in the usual unitary model of quantum computation. Those problems (such as measurements, or noise and decoherence) disappear in the non-unitary (i.e. non-reversible) model. In fact, although quantum computational processes permit measurements in the middle of the computation, however, the state of the computation after a measurement is a mixed state. Moreover, to implement quantum computers, noise and, in particular, decoherence are important obstacles. The main problem in this interface between quantum physics and quantum computation models is that quantum noise and decoherence are non-unitary operations that cause a pure state to become a mixed state.
In this note, we propose a simple construction on density matrices (dubbed *polynomial operations*) that permits to resort, in terms of probability distributions, to a Stone-Weierstrass type theorem. That result implies that any continuous function can be regarded, from a probabilistic point of view, and up to a certain approximation, as a quantum operation, Theorems 3..3 and 3..4. The results are organized as follows: in Section 2. we provide all the basic notions, in Section 3. we show an overview of a probabilistic version of the Stone-Weierstrass theorem in the framework of quantum operations (further detalis are showed in [11]), in Section 4. some applications of our main result to the case of product MV-algebra [28] are given, and lastly in Section 5. some possible future

investigation issues are illustrated and a few conclusive remarks drawn.

2. Basic notions

A quantum system in a pure state is described by a unit vector in a Hilbert space. In the Dirac notation a pure state is denoted by $|\psi\rangle$ ($\langle\psi|$). A *quantum bit* or *qubit*, the fundamental concept of quantum computation, is a pure state in the Hilbert space $\mathbb{C}^2$. The standard orthonormal basis $\{|0\rangle, |1\rangle\}$ of $\mathbb{C}^2$ is called the *logical basis*. Thus a qubit $|\psi\rangle$ may be written as a linear superposition of the basis vectors with complex coefficients $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$ with $|c_0|^2 + |c_1|^2 = 1$.

Quantum mechanics reads out the information content of a pure state via the Born rule, according to which the probability value assigned to a qubit is defined as follows:

$$\mathtt{p}(|\psi\rangle) = |c_1|^2.$$

The states of interest for quantum computation lie in the tensor product $\otimes^n\mathbb{C}^2 = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$, where $\otimes^n\mathbb{C}^2 = \mathbb{C}^2$ if $n = 1$. The space $\otimes^n\mathbb{C}^2$ is a 2^n-dimensional complex Hilbert space. The 2^n-*computational basis* consists of the 2^n orthogonal states $|\iota\rangle$ $(0 \leq \iota \leq 2^n)$, where ι is in binary representation and it can be seen as the tensor product of the states $|\iota_1\rangle \otimes |\iota_2\rangle \otimes \cdots \otimes |\iota_n\rangle$ where $\iota_j \in \{0,1\}$. A pure state $|\psi\rangle \in \otimes^n\mathbb{C}^2$ is generally a superposition of the basis vectors: $|\psi\rangle = \sum_{\iota=1}^{2^n} c_\iota|\iota\rangle$ with $\sum_{\iota=1}^{2^n} |c_\iota|^2 = 1$.

In general, a quantum system is not in a pure state. This might be because the system is coupled with an environment, it is subject to a measurement process etc. In those cases, the state-evolution is no longer reversible and the system is said to be in a *mixed state*. A convenient mathematical description of a mixed state is given by the notion of *density operator*, i.e. an Hermitian positive operator ρ on a 2^n-dimensional complex Hilbert space with trace $\mathtt{tr}(\rho) = 1$.

A pure state $|\psi\rangle$ can be represented as a limit case of mixed state in the form $\rho = |\psi\rangle\langle\psi|$. In particular, each vector of the logical basis of $\mathbb{C}^2$ can be associated to a density operator $P_0 := |0\rangle\langle 0|$ or $P_1 := |1\rangle\langle 1|$ that represents the *falsity-property* and the *truth-property*, respectively. One can represent an arbitrary density matrix ρ in terms of a tensor products of the Pauli matrices:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

in the following way:

$$\rho = \frac{1}{2^n} \sum_{\mu_1 \ldots \mu_n} P_{\mu_1 \ldots \mu_n} \sigma_{\mu_1} \otimes \cdots \otimes \sigma_{\mu_n},$$

where $\mu_i \in \{0, x, y, x\}$ for each $i \in \{1, \ldots, n\}$. The real expansion coefficients $P_{\mu_1 \ldots \mu_n}$ are given by $P_{\mu_1 \ldots \mu_n} = \mathtt{tr}(\sigma_{\mu_1} \otimes \cdots \otimes \sigma_{\mu_n} \rho)$. Since the eigenvalues of the Pauli matrices are ± 1, the expansion coefficients satisfy the inequality $|P_{\mu_1 \ldots \mu_n}| \leq 1$. In what follows, for sake of simplicity, we will use without distinction I or σ_0. We denote by $\mathscr{D}(\otimes^n \mathbb{C}^2)$ the set of all density operators of $\otimes^n \mathbb{C}^2$; hence the set $\mathscr{D} = \bigcup_{i \in N} \mathscr{D}(\otimes^n \mathbb{C}^2)$ will be the set of all possible density operators. Moreover, we can identify in each space $\mathscr{D}(\otimes^n \mathbb{C}^2)$ two special operators $P_0^{(n)} = \frac{1}{2^n} I^{n-1} \otimes P_0$ and $P_1^{(n)} = \frac{1}{2^n} I^{n-1} \otimes P_1$ that represent, in this framework, the falsity-property and the truth-property, respectively. The *probability of truth* $\mathtt{p}$ of a density operator ρ is dictated by the Born rule and equals

$$\mathtt{p}(\rho) = \mathtt{tr}(P_1^{(n)} \rho).$$

In case $\rho = |\psi\rangle\langle\psi|$, where $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$, then $\mathtt{p}(\rho) = |c_1|^2$.
Let $\rho \in \mathscr{D}(\mathbb{C}^2)$. Then ρ can be represented as a linear superposition $\rho = \frac{1}{2}(I + r_x\sigma_x + r_y\sigma_y + r_z\sigma_z)$, where r_x, r_y, r_z are real numbers such that $r_x^2 + r_y^2 + r_z^2 \leq 1$. Therefore, every density operator ρ in $\mathscr{D}(\mathbb{C}^2)$ has the matrix representation:

$$\rho = \frac{1}{2} \begin{pmatrix} 1 + r_z & r_x - ir_y \\ r_x + ir_y & 1 - r_z \end{pmatrix} = \begin{pmatrix} 1 - \alpha & \beta \\ \beta^* & \alpha \end{pmatrix} \tag{1}$$

Furthermore, any real number λ $(0 \leq \lambda \leq 1)$ uniquely determines a density operator as follows:

$$\rho_\lambda = (1 - \lambda)P_0 + \lambda P_1 = \frac{1}{2}(I + (1 - 2\lambda)\sigma_z) = \begin{pmatrix} 1 - \lambda & 0 \\ 0 & \lambda \end{pmatrix} \tag{2}$$

In virtue of (1) and (2), one may verify that, whenever $\rho \in \mathscr{D}(\mathbb{C}^2)$, then $\mathtt{p}(\rho) = \frac{1-r_z}{2}$ and $\mathtt{p}(\rho_\lambda) = \lambda$. Thus each density operator ρ in $\mathscr{D}(\mathbb{C}^2)$ can be written as

$$\rho = \begin{pmatrix} 1-\mathtt{p}(\rho) & a \\ a^* & \mathtt{p}(\rho) \end{pmatrix} \tag{3}$$

In the usual model of quantum computation the state of a system is pure and the operations (*quantum gates*) are represented by unitary operators. Nevertheless, in case a system is not completely isolated from the environment its evolution is, in general, irreversible. A model of quantum computing that relates to that phenomenon is mathematically described by means of *quantum operations* (as quantum gates) acting on density operators (as information quantities).
Given a finite dimensional complex Hilbert space H, we will denote by $\mathscr{L}(H)$ the vector space of all linear operators on H. Let H_1, H_2 be two finite dimensional complex Hilbert spaces. A *super operator* is a linear operator $\mathscr{E}$: $\mathscr{L}(H_1) \to \mathscr{L}(H_2)$ sending density operators to density operators [2]. This is equivalent to say that $\mathscr{E}$ is trace-preserving and positive, i.e. sends positive semi-definite Hermitian operators to positive semi-definite Hermitian operators. A super operator $\mathscr{E}$ is said to be a *quantum operation* iff the super operator $\mathscr{E} \otimes I_H$ is positive, where I_H is the identity super operator on an arbitrary finite dimensional complex Hilbert space H. In this case $\mathscr{E}$ is also called *completely positive*. The following theorem, dued to K. Kraus [24], provide an equivalent definition of quantum operations:

Theorem 2..1. *A linear operator $\mathscr{E} : \mathscr{L}(H_1) \to \mathscr{L}(H_2)$ is a quantum operation iff $\forall \rho \in \mathscr{L}(H_1)$:*

$$\mathscr{E}(\rho) = \sum_i A_i \rho A_i^\dagger$$

for some set of operators $\{A_i\}$ such that $\sum_i A_i^\dagger A_i = I$.

3. A probabilistic Stone-Weierstrass type theorem

The aim of the present section is to propose a representation, in probabilistic terms, of a particular class of polynomials via quantum operations. Such a result will be expedient to prove a probabilistic Stone-Weierstrass type theorem. First of all, let us introduce some notations and preliminary definitions. The term *multi-index* denotes an ordered n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ of non negative integers

α_i. The *order* of α is given by $|\alpha| = \alpha_1 + \ldots\ldots + \alpha_n$. If $\mathbf{x} = (x_1,\ldots,x_n)$ is an n-tuple of variables and $\alpha = (\alpha_1,\ldots,\alpha_n)$ a multi-index, the monomial $\mathbf{x}^\alpha$ is defined by $\mathbf{x}^\alpha = x_1^{\alpha_1}x_2^{\alpha_2}\ldots x_n^{\alpha_n}$. In this language a real polynomial of order k is a function $P(\mathbf{x}) = \sum_{|\alpha|\leq k} a_\alpha \mathbf{x}^\alpha$ such that $a_\alpha \in \mathbb{R}$.
Let $\mathbf{x} = (x_1,\ldots,x_n)$ and k be a natural number. If we define the set $D_k(\mathbf{x})$ as follows:

$$D_k(\mathbf{x}) = \{(1-x_1)^{\alpha_1}x_1^{\beta_1}\ldots(1-x_n)^{\alpha_n}x_n^{\beta_n} : \alpha_i+\beta_i = k,\ \ i \in \{1,\ldots,n\}\}, \qquad (4)$$

then we obtain the following useful lemmas:

Lemma 3..1. *Let $\mathbf{X}_1,\ldots,\mathbf{X}_n$ be a family of matrices such that*

$$\mathbf{X}_i = \begin{pmatrix} 1-x_i & b_i \\ b_i^* & x_i \end{pmatrix}$$

and let $\mathbf{X} = (\otimes^k\mathbf{X}_1)\otimes(\otimes^k\mathbf{X}_2)\otimes\cdots\otimes(\otimes^k\mathbf{X}_n)$. Then

$$Diag(\mathbf{X}) = D_k(x_1,\ldots,x_n).$$

Proof. It can be verified that $\otimes^k X_i = \{h_1h_2\ldots h_k : h_j \in \{(1-x_i),x_i\}, 1\leq j\leq k\} = \{(1-x_1)^\alpha x_1^\beta : \alpha+\beta = k\}$. Thus, $(\otimes^k\mathbf{X}_1)\otimes(\otimes^k\mathbf{X}_2)\otimes\cdots\otimes(\otimes^k\mathbf{X}_n) = \{(1-x_1)^{\alpha_1}x_1^{\beta_1}\ldots(1-x_n)^{\alpha_n}x_n^{\beta_n} : \alpha_i+\beta_i = k,\ \ i\in\{1,\ldots,n\}\}$. Whence our claim follows. $\square$

Lemma 3..2. *Let $\mathbf{x} = (x_1,\ldots,x_n)$ and k be a natural number. For any monomial $\mathbf{x}^\alpha$, such that $\mid\alpha\mid\leq k$, the following conditions hold:*

1. $\mathbf{x}^\alpha = \sum_{\mathbf{y}\in D_k(\mathbf{x})} \delta_\mathbf{y}\mathbf{y}$;
2. $1-\mathbf{x}^\alpha = \sum_{\mathbf{y}\in D_k(\mathbf{x})} \gamma_y\mathbf{y}$;

   where δ_y and γ_y are in $\{0,1\}$.

Proof. First, let us define, for any $i\in\{1,\ldots,n\}$, a matrix $\mathbf{X_i}$ as follows

$$\mathbf{X}_i = \begin{pmatrix} 1-x_i & 0 \\ 0 & x_i \end{pmatrix}$$

1) Let $\mathbf{x}^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$ such that $\mid \alpha \mid \leq k$. Thus, there are $s_1, \dots, s_n$ such that $\alpha_i + s_i = k$. Let $\mathbf{W} = (\otimes^{s_1} \mathbf{X}_1) \otimes (\otimes^{s_2} \mathbf{X}_2) \otimes \dots \otimes (\otimes^{s_n} \mathbf{X}_n)$ and consider the matrix $\mathbf{W}\mathbf{x}^\alpha$. In view of Lemma 3..1, $Diag(\mathbf{W}\mathbf{x}^\alpha) \subseteq D_k(x_1, \dots, x_n)$ since every element in $Diag(\mathbf{W}\mathbf{x}^\alpha)$ is a monomial of order nk. Further, since $\mathtt{tr}(\mathbf{W}\mathbf{x}^\alpha) = (\mathtt{tr}\mathbf{W})\mathbf{x}^\alpha = 1\mathbf{x}^\alpha = \mathbf{x}^\alpha$, we obtain $\mathbf{x}^\alpha = \mathtt{tr}(\mathbf{W}\mathbf{x}^\alpha)$, i.e. the required polynomial expansion.
2) Let $\mathbf{X} = (\otimes^k \mathbf{X}_1) \otimes (\otimes^k \mathbf{X}_2) \otimes \dots \otimes (\otimes^k \mathbf{X}_n)$. By Lemma 3..1 $Diag(\mathbf{X}) = D_k(x_1, \dots, x_n)$ and $\mathtt{tr}(\mathbf{X}) = 1$. Upon recalling that $\mathbf{x}^\alpha = \sum_{\mathbf{y} \in D_k(\mathbf{x})} \delta_\mathbf{y} \mathbf{y}$, we define $\gamma_\mathbf{y} = 1$ if $\delta_\mathbf{y} = 0$ and $\gamma_\mathbf{y} = 0$ if $\delta_\mathbf{y} = 1$. Therefore,

$$\begin{aligned} 1 &= \mathtt{tr}(\mathbf{X}) \\ &= \sum_{\mathbf{y} \in D_k(\mathbf{x})} \delta_\mathbf{y} \mathbf{y} + \sum_{\mathbf{y} \in D_k(\mathbf{x})} \gamma_y \mathbf{y} \\ &= \mathbf{x}^\alpha + \sum_{\mathbf{y} \in D_k(\mathbf{x})} \gamma_y \mathbf{y} \end{aligned}$$

and $1 - \mathbf{x}^\alpha = \sum_{\mathbf{y} \in D_k(\mathbf{x})} \gamma_y \mathbf{y}$

□

In virtue of the previous claims, we can prove a technical but rather important theorem:

Theorem 3..3. *Let $\mathbf{x} = (x_1, \dots, x_n)$ be an n-tuple of variables, and let $P(\mathbf{x}) = \sum_{\mathbf{y} \in D_k(\mathbf{x})} a_\mathbf{y} \mathbf{y}$ be a polynomial such that $\mathbf{y} \in D_k(\mathbf{x})$, $0 \leq a_\mathbf{y} \leq 1$ and the restriction $P(\mathbf{x}) \upharpoonright_{[0,1]^n}$ be such that $0 \leq P(\mathbf{x}) \upharpoonright_{[0,1]^n} \leq 1$.*
Then there exists a polynomial quantum operation $\mathscr{P} : \mathscr{L}(\otimes^{nk} \mathbb{C}^2) \to \mathscr{L}(\otimes^{nk} \mathbb{C}^2)$ such that, for any n-tuple $\sigma = (\sigma_1, \dots, \sigma_n)$ in $\mathscr{D}(\mathbb{C}^2)$,

$$\mathtt{p}(\mathscr{P}((\otimes^k \sigma_1) \otimes \dots \otimes (\otimes^k \sigma_n))) = P(\mathtt{p}(\sigma_1), \dots, \mathtt{p}(\sigma_n)).$$

Moreover,

$$\mathscr{P}((\otimes^k \sigma_1) \otimes \dots \otimes (\otimes^k \sigma_n)) = (\frac{1}{2^{nk-1}} \otimes^{nk-1} I) \otimes \rho_{P(\mathtt{p}(\sigma_1), \dots, \mathtt{p}(\sigma_n))}.$$

Proof. Let $\sigma_1, \ldots, \sigma_n$ be density operators on $\mathbb{C}^2$. Assume that for any σ_i

$$\sigma_i = \begin{pmatrix} 1-x_i & b_i \\ b_i^* & x_i \end{pmatrix}$$

Hence, $\mathrm{p}(\sigma_i) = x_i$. Evidently, $\sigma = (\otimes^k \sigma_1) \otimes \cdots \otimes (\otimes^k \sigma_n)$ is a $2^{nk} \times 2^{nk}$ matrix and, by Lemma 3..1, $Diag(\sigma) = D_k(x_1, \ldots, x_n)$. Thus, each $\mathbf{y} \in D_k(\mathbf{x})$ can be seen as the (i,i)-th entry of $Diag(\sigma)$. Further, the polynomial $P(\mathbf{x}) = \sum_{\mathbf{y} \in D_k(\mathbf{x})} a_{\mathbf{y}} \mathbf{y} = \sum_{j=1}^{2^{nk}} a_j \mathbf{y}_j$ is such that every $\mathbf{y}_j$ is the (j,j)-th entry of $Diag(\sigma)$. Let, now, $\mathbf{y}_{j_0} \in Diag(\sigma)$.

a) We want to place the elements of the form $a_{j_0} \mathbf{y}_{j_0}$ in the $(2s, 2s)$-th entries of a $2^{nk} \times 2^{nk}$ matrix.
Let us consider the $2^{nk} \times 2^{nk}$ matrix $A_{j_0}^{2s} = \sqrt{\frac{a_{j_0}}{2^{nk-1}}} D_{j_0}^{2s}$ such that $D_{j_0}^{2s}$ has 1 just in the $(2s, j_0)$-th entry and 0 in any other entry. One may verify that $A_{j_0}^{2s} \sigma (A_{j_0}^{2s})^\dagger$ is the required matrix. Moreover:

$$\sum_{2s} A_{j_0}^{2s} \sigma (A_{j_0}^{2s})^\dagger = \frac{1}{2^{nk-1}} \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & a_{j_0}\mathbf{y}_{j_0} & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & a_{j_0}\mathbf{y}_{j_0} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

b) Let us recall that $1 = \sum_{\mathbf{y} \in D_k(\mathbf{x})} \mathbf{y} = \sum_{j=1}^{2^{nk}} \mathbf{y}_j$. Then:

$$1 - \sum_{j=1}^{2^{nk}} a_j \mathbf{y}_j = \sum_{j=1}^{2^{nk}} \mathbf{y}_j - \sum_{j=1}^{2^{nk}} a_j \mathbf{y}_j = \sum_{j=1}^{2^{nk}} (1-a_j) \mathbf{y}_j.$$

We now want to stick the elements of the form $(1-a_{j_0})\mathbf{y}_{j_0}$ into the $(2s-1, 2s-1)$-th entries of a $2^{nk} \times 2^{nk}$ matrix. Let us consider the $2^{nk} \times 2^{nk}$ matrix $A_{j_0}^{2s-1} = \sqrt{\frac{1-a_{j_0}}{2^{nk-1}}} D_{j_0}^{2s-1}$ such that $D_{j_0}^{2s-1}$ have 1 just in the $(2s-1, j_0)$-th entry and 0 in any other entry. Again, one may verify that $A_{j_0}^{2s-1} \sigma (A_{j_0}^{2s-1})^\dagger$ is the required matrix. Furthermore:

$$\sum_{2s-1} A_{j_0}^{2s-1}\sigma(A_{j_0}^{2s-1})^\dagger = \frac{1}{2^{nk-1}}\begin{pmatrix} (1-a_{j_0})\mathbf{y}_{j_0} & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & (1-a_{j_0})\mathbf{y}_{j_0} & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Thus, some calculations show that:

$$\begin{aligned} \mathscr{P} &= \sum_{j_0}\sum_{2s} A_{j_0}^{2s}\sigma(A_{j_0}^{2s})^\dagger + \sum_{j_0}\sum_{2s-1} A_{j_0}^{2s-1}\sigma(A_{j_0}^{2s-1})^\dagger \\ &= (\frac{1}{2^{kn-1}}\otimes^{nk-1} I)\otimes\begin{pmatrix} 1-\sum_{j=1}^{2^{nk}} a_j\mathbf{y}_j & 0 \\ 0 & \sum_{j=1}^{2^{nk}} a_j\mathbf{y}_j \end{pmatrix} \end{aligned}$$

Now, set $A = \sum_{j_0}\sum_{2s}(A_{j_0}^{2s})^\dagger A_{j_0}^{2s} + \sum_{j_0}\sum_{2s+1}(A_{j_0}^{2s+1})^\dagger A_{j_0}^{2s+1}$. Our task is to verify that $A = I$.

c) First of all, notice that the matrix $(A_{j_0}^{2s})^\dagger A_{j_0}^{2s}$ has the value $\frac{a_{j_0}}{2^{nk-1}}$ just in the (j_0, j_0)-th entry, while any other entry is 0. Therefore, the matrix $\sum_{2s}(A_{j_0}^{2s})^\dagger A_{j_0}^{2s}$ has the value $\frac{2^{nk-1}a_{j_0}}{2^{nk-1}} = a_{j_0}$ in the (j_0, j_0)-th entry and all the other entries equal 0. Hence:

$$\sum_{j_0}\sum_{2s}(A_{j_0}^{2s})^\dagger A_{j_0}^{2s} = \begin{pmatrix} a_1 & 0 & 0 & 0 & \dots \\ 0 & a_2 & 0 & 0 & \dots \\ 0 & 0 & a_3 & 0 & \dots \\ 0 & 0 & 0 & a_4 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

d) On the other hand the matrix $(A_{j_0}^{2s-1})^\dagger A_{j_0}^{2s-1}$ has the value $\frac{1-a_{j_0}}{2^{nk-1}}$ just in the (j_0, j_0)-th entry and 0 in any other. Therefore, the matrix $\sum_{2s-1}(A_{j_0}^{2s-1})^\dagger A_{j_0}^{2s-1}$ has the value $\frac{2^{nk-1}(1-a_{j_0})}{2^{nk-1}} = 1-a_{j_0}$ in the (j_0, j_0)-th entry and any other is 0.

Hence:

$$\sum_{j_0}\sum_{2s-1}(A^{2s-1}_{j_0})^{\dagger}A^{2s-1}_{j_0} = \begin{pmatrix} 1-a_1 & 0 & 0 & 0 & \dots \\ 0 & 1-a_2 & 0 & 0 & \dots \\ 0 & 0 & 1-a_3 & 0 & \dots \\ 0 & 0 & 0 & 1-a_4 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Thus, $\sum_{j_0}\sum_{2s}(A^{2s}_{j_0})^{\dagger}A^{2s-1}_{j_0} + \sum_{j_0}\sum_{2s-1}(A^{2s-1}_{j_0})^{\dagger}A^{2s-1}_{j_0} = I$. Whence, by Theorem 2..1, $\mathscr{P}$ is a quantum operation. $\square$

In virtue of Theorem 3..3, we can now establish a probabilistic version of the classical Stone-Weierstrass theorem.

Theorem 3..4. *Let* $\mathbf{x} = (x_1,\dots,x_n)$ *be an n-tuple of variables and* $f : [0,1]^n \to (0,1)$ *be a continuous function. Then, for each* $\varepsilon > 0$ *there exists a quantum operation* $\mathscr{P} : \mathscr{L}(\otimes^{nk}\mathbb{C}^2) \to \mathscr{L}(\otimes^{nk}\mathbb{C}^2)$ *and a constant* $M \geq 1$ *such that, for any density matrix* $\sigma = (\otimes^k \sigma_1)\otimes\cdots\otimes(\otimes^k \sigma_n)$*, the following inequality holds:*

$$\mid \mathtt{p}(\mathscr{P}(\sigma)) - \frac{1}{M}f(\mathtt{p}(\sigma_1),\dots,\mathtt{p}(\sigma_n)) \mid \leq \varepsilon.$$

Proof. Let $f : [0,1]^n \to (0,1)$ be a continuous function. By the classical Stone-Weierstrass theorem, there exists a polynomial $P_0(x_1,\dots,x_n) = \sum_{|\alpha|\leq k} a_\alpha \mathbf{x}^\alpha$ such that for each $\varepsilon > 0$, $\mid P_0 - f \mid \leq \frac{\varepsilon}{2}$. Let $a_{\alpha_1},\dots,a_{\alpha_n}$ be positive coefficients and $a_{\beta_1},\dots,a_{\beta_s}$ be negative coefficients in the polynomial $P_0(x_1,\dots,x_n)$. Let M be a positive real number such that $\frac{\sum_{i=1}^{s}|a_{\beta_i}|}{M} \leq \frac{\varepsilon}{2}$. Let us define a polynomial P by $P(x_1,\dots,x_n) = \sum_{i=1}^{n}\frac{a_{\alpha_i}}{M}\mathbf{x}^{\alpha_i} + \sum_{j=1}^{s}\frac{|a_{\beta_j}|}{M}(1-\mathbf{x}^{\beta_j})$. Then, we obtain that $P(x_1,\dots,x_n) = \frac{1}{M}P_0(x_1,\dots,x_n) + \frac{\sum_{i=1}^{s}|a_{\beta_i}|}{M}$. Therefore, in $[0,1]^n$:

$$\begin{aligned} \mid P - \frac{1}{M}f \mid &= \mid P - \frac{1}{M}P_0 + \frac{1}{M}P_0 - \frac{1}{M}f \mid \\ &\leq \mid P - \frac{1}{M}P_0 \mid + \mid \frac{1}{M}P_0 - \frac{1}{M}f \mid \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{M} \\ &\leq \varepsilon \end{aligned}$$

So, by Lemma 3..2, we obtain that $P(\mathbf{x}) = \sum_{\mathbf{y}\in D_k(\mathbf{x})} a_{\mathbf{y}}\mathbf{y}$ is such that $a_{\mathbf{y}} \geq 0$ and $0 \leq P(\mathbf{x}) \leq 1$ in $[0,1]^n$. Whence, by Theorem 3..3, there exists a quantum operation $\mathscr{P} : \mathscr{L}(\otimes^{nk}\mathbb{C}^2) \to \mathscr{L}(\otimes^{nk}\mathbb{C}^2)$ associated to P such that for each n-tuple $\sigma = (\sigma_1,\ldots,\sigma_n)$, with σ_i in $\mathscr{D}(\mathbb{C}^2)$, $\mathtt{p}(\mathscr{P}((\otimes^k\sigma_1)\otimes\cdots\otimes(\otimes^k\sigma_n))) = P(x_1/\mathtt{p}(\sigma_1),\ldots,x_n/\mathtt{p}(\sigma_n))$.[1] Thus

$$\mid \mathtt{p}(\mathscr{P}((\otimes^k\sigma_1)\otimes\cdots\otimes(\otimes^k\sigma_n)))) - \frac{1}{M} f(\mathtt{p}(\sigma_1),\ldots,\mathtt{p}(\sigma_n)) \mid \leq \varepsilon.$$

□

4. Representing the standard PMV-operations

In this section we apply the results obtained to two functions (namely, the product t-norm $\bullet$, and the Łukasiewicz conorm $\oplus$) of definite importance in fuzzy logic. Let us recall some notions first.
The *standard PMV-algebra* [10, 28] is the algebra

$$[0,1]_{PMV} = \langle [0,1], \oplus, \bullet, \neg, 0, 1\rangle,$$

where $[0,1]$ is the real unit segment, $x \oplus y = \min(1, x+y)$, $\bullet$ is the real product, and $\neg x = 1 - x$. This structure plays a notable role in quantum computing, in that it decribes, in a probabilistic way, a relevant system of quantum gates named *Poincarè irreversible quantum computational algebra* [5, 8]. The connection between quantum computational logic with mixed states and fuzzy logic, comes from the election of a system of quantum gates such that, when interpreted under probabilistic semantics, they turn out in some kind of operation in the real interval $[0,1]$. The above-mentionated $PMV-algebra$ is a structure that represents algebraic counterpart of the probabilistic semantics conceived from the continuous t-norm. On the other hand, the use of fuzzy logics (and infinite-valued Łukasiewicz logic in particular) in game theory and theoretical physics was pioneered in [26, 27], linking the mentioned structures with Ulam games and $AF - C^*$-algebras, respectively. We will pay special attention to the study of the Łukasiewicz t-norm, due to its relation with Ulam games and its

[1] By $x_i/\mathtt{p}(\sigma_i)$ we mean the attribution of the value $\mathtt{p}(\sigma_i)$ to the variable x_i.

possible applications to error-correction codes in the context of quantum computation.
Evidently, $\neg$ can be expressed as a polynomial in the generator system $D_1(x)$; whence by Theorem 3..3, it is representable as a polynomial quantum operation. A possible representation can be the following: $NOT(\rho) = \sigma_x \rho \sigma_x^\dagger$. In fact, $\mathtt{p}(NOT(\rho)) = 1 - \mathtt{p}(\rho)$.
Furthermore, $\bullet$ can be represented by a polynomial in the generator system $D_2(x,y)$. According with the construction presented in Theorem 3..3, the following representation obtains. Let us consider the following matrices:

$$G_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad G_2 = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad G_3 = \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$G_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad G_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad G_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$G_7 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \quad G_8 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

One may verify that $\sum_{i=1}^{8} G_i(\tau \otimes \sigma)G_i^\dagger = \frac{1}{2} I \otimes \rho_{\mathtt{p}(\tau)\mathtt{p}(\sigma)}$ where $\sigma, \tau \in \mathscr{D}(\mathbb{C}^2)$. Thus, by Kraus representation Theorem [24], $\sum_{i=1}^{8} G_i(\tau \otimes \sigma)G_i^\dagger$ is a quantum operation, and $\mathtt{p}(\sum_{i=1}^{8} G_i(\tau \otimes \sigma)G_i^\dagger) = \mathtt{p}(\tau) \bullet \mathtt{p}(\sigma)$. That quantum operation represents the well known quantum gate *IAND* modulo a tensor power [5, 29].
As regards the Łukasiewicz conorm $\oplus$, it can be seen that it is not a polynomial, Figure 1.
Therefore, our idea is to obtain a polynomial $P(x,y)$ in some generator system $D_k(x,y)$, such that $P(x,y)$ can approximate the Łukasiewicz sum.
By using numerical methods, we get the following approximating polynomial of $\oplus$ in $[0,1]$:

$$P(x,y) = \frac{5}{12}x(1-x) + \frac{5}{12}y(1-x) + \frac{5}{12}x(1-y) + \frac{5}{12}y(1-y) + \frac{1}{2}x + \frac{1}{2}y,$$

whose graph is depicted in Figure 2.

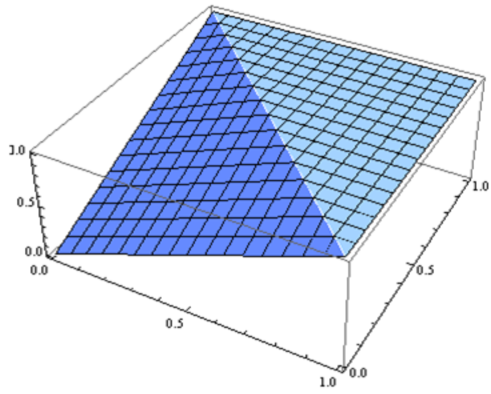

Figure 1: The Łukasiewicz conorm

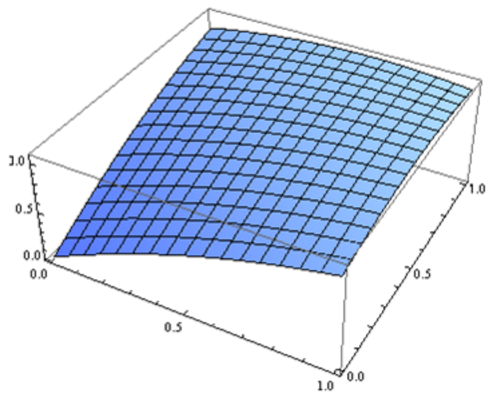

Figure 2: $P(x,y)$

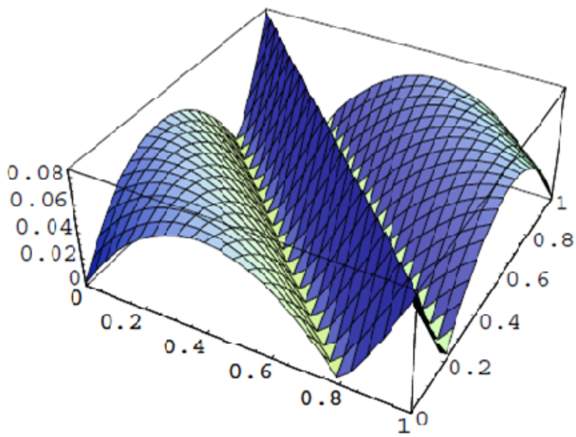

Figure 3: $(x \oplus y) - P(x,y)$

Let us remark that $0 \leq P(x,y) \leq x \oplus y$. Therefore, $e = \max_{[0,1]}\{(x \oplus y) - P(x,y)\} \leq 0.08$, as Figure 3 shows. Furthermore, one readily realizes that $P(x,y)$ is a polynomial obtained from the generator system $D_2(x,y)$, that also satisfies the hypothesis of Theorem 3..3. Thus, $P(x,y)$ is representable as a polynomial quantum operation $\mathscr{P}_\oplus$, where

$$\mathtt{p}(\mathscr{P}_\oplus(\tau \otimes \sigma)) = (\mathtt{p}(\tau) \oplus \mathtt{p}(\sigma)) \pm 0.08$$

5. Conclusions and Open problems

In virtue of the results in Section 4., it turns out that the approximation obtained in the case of PMV-algebras is definitely accurate. Further, in [11], authors show a covergence theorem that allows to achieve every degree of accuracy; the price to pay is the increasing of the degree of the approximating polynomial. In our opinion, that is an interesting achievement, since it provides a (quantum) computational motivation for the investigation of algebraic structures equipped with the Łukasiewicz sum and, to a certain extent, it relates "classical" fuzzy

logic to quantum computational logics.
Nonetheless, some general remarks on the whole construction are in order as well.

1. If one wants to apply our results to the models of (quantum) computing, efficiency is of central importance. Unfortunately, since the number of copies required in our construction corresponds to the degree of the approximating polynomial, it is impossible to generally specify the dimension of the Hilbert space required to achieve, given a certain ε, the approximating polynomial.

2. Since the work of Ekert and other scholars [22, 20, 19, 9], a direct study of estimations of linear and non-linear functionals of (quantum) states using quantum networks has been proposed. This approach has the advantage that it bypasses quantum tomography, providing more direct estimations of both linear and non-linear functionals of a state. It could be of interest, in our opinion, to investigate if, and in what cases, our construction can be carried out by a quantum network.

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