RESEARCH ARTICLE

Some generalizations of fuzzy structures in quantum computational logic

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Quantum computational logics provide a fertile common ground for a unified treatment of vagueness and uncertainty. In this paper we describe an approach to the logic of quantum computation that has been recently taken up and developed by the present authors. Special attention will be devoted to a generalisation of Chang's MV algebras (called *quasi-MV algebra*) which abstracts over the algebra whose universe is the set of qumixes of the 2-dimensional complex Hilbert space, as well as to its expansions by additional quantum connectives. We furthermore explore some future research perspectives, also in the light of some recent limitative results whose general significance will be duly assessed.

1. Introduction

Fuzzy logic is normally associated to an attempt to provide a rigorous treatment of the *vagueness* phenomenon; the expression "quantum theory", on the other hand, reminds even the layman of such things as the *uncertainty* principle. Vagueness and uncertainty, at least at first sight, seem to have little in common. Uncertainty is an *epistemic* phenomenon by its very definition: a predicate is uncertain (relative to a specified knower A) in case the information in A's possession is not sufficient to determine its applicability in a given case. Uncertainty, in other words, is a matter of ignorance; "carnivore" can be uncertain as a predicate of ornithorhynchuses for a subject who doesn't know whether or not ornithorhynchuses are, indeed, carnivores. On the contrary, the status of *vagueness* is a prototypically contentious matter. Although only a handful of authors would dispute that a predicate is vague just in case it admits of borderline cases of application, specialists are notoriously at variance as to whether such unfitness to sharp cut-off points depends on meaning, or on knowledge, or else on things in themselves. Supervaluationists like Fine (15) take vagueness to be a form of ambiguity, while epistemicists like Williamson (29) prefer to consider it a form of ignorance. There is no shortage of pragmatic (e.g. (27)) or ontological (e.g. (20)) theories either, so that nearly anybody can find her own favorite approach available on the market.

Once we enter the quantum world, however, things become more complicated than the above-sketched picture would suggest. Here, we can no longer draw clear boundary lines between the territories of vagueness and uncertainty. Any superposition pure state represents a maximal knowledge of the observer - a piece of information that cannot be consistently extended to a richer information - but does not,

in general, semantically decide all events that may hold for it[1]; any mixed state, on the contrary, represents a non-maximal piece of information. In the latter case an evident epistemic feature comes into play; in the former, however, are we to talk of *uncertainty* - given that the applicability of a predicate can be determined, in general, only up to a probability assignment - or of *vagueness* - since the epistemic dimension of ignorance is clearly not the issue whenever we are concerned with a maximal information?

According to the *unsharp approach* to quantum theory (see e.g. (23)), we ought to consider this dimension of *ontological uncertainty* as a brand new category which coexists alongside the ones we mentioned above, calling for its own specific mathematical treatment in much the same way as classical probability theory turns out to be the proper formal framework for classical uncertainty, or in much the same way as fuzzy logic or any of its rivals (e.g. supervaluation theory) affords a rigorous grip of vagueness. In standard (sharp) quantum logic *à la* Birkhoff-von Neumann, propositions ascribing properties are represented by projection operators (or, equivalently, by closed subspaces of a Hilbert space). It follows that all properties are necessarily not vague: the possible values of a given physical quantity are expressed by the eigenvalues of the corresponding self-adjoint operator,

and projection operators have eigenvalues in $\{0, 1\}$ - meaning that either the property at issue definitely holds or it definitely does not hold. In unsharp quantum theory and in unsharp quantum logic, however, a more general notion of property has been suggested. Projections are replaced there by *effects*, whose eigenvalues may range throughout the whole real interval $[0, 1]$. Unsharp quantum theory, therefore, accommodates "vague" properties as well, which are not an all-or-nothing matter but may hold to a given degree. True to form, the mathematical structures that arise within this research stream are, more often than not, either closely related to fuzzy logical structures or even plain generalizations of such (see e.g. (18), (16), (6)).

Quantum computational logic, as developed by Maria Luisa Dalla Chiara, Gianpiero Cattaneo and other authors, including the present writers ((4), (5), (10), (11), (12)), departs even more drastically from the standard Birkhoff-von Neumann approach. Meanings of sentences are no longer formalized through closed subspaces of a Hilbert space, but by means of *quantum information units*: qubits, quregisters, qumixes (see the next section). Fuzzy-like structures, however, appear in this setting, too. The aim of the present survey is to clarify in detail this further bridge between fuzzy logic and quantum logic. For detailed proofs of the results stated in this paper, the reader is referred to the work cited in the bibliography. Although we tried to make the paper self-contained and accessible to readers with no background in quantum computation, some previous acquaintance with the subject may be useful; the reader can consult e.g. (25).

2.   A primer of quantum computation

At the very beginning of the XX century, quantum mechanics and computation theory were two fundamental theories studied in completely separated ways. Subsequently, the increasing miniaturization of the hardware parts of computing devices and the strenuous attempts to increase computational efficiency demanded a new idea of computation.

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[1]Meaning that there might be a sentence α, such that neither α nor its negation $\neg\alpha$ holds for the state at issue.

The first author who envisaged an application of quantum mechanics to computation theory was Richard Feynman. He demonstrated (see e.g. (25)) that no Turing machine could ever simulate some physical systems without incurring into an exponential performance slowdown, while an universal quantum simulator would perform far more efficiently. After the seminal work of David Deutsch, who provided in 1985 ((14)) the first mathematical framework for the so called *universal quantum Turing machine*, the literature concerning what we can call the quantum approach to computation has enormously increased, bringing to what, nowadays, is a science of its own: *quantum computation.*

2.1 *Qubits and superposition states*

It is well-known that in quantum mechanics a physical system is naturally associated to a Hilbert space. We say that a *state*, as given by a unit vector in such a Hilbert space (see e.g. (9)), is *pure* if and only if it represents a maximal information quantity, i.e. a piece of information on the physical system that could not be consistently augmented by any further observation.

Consider the two-dimensional Hilbert space $\mathbb{C}^2$, and let $\{|0\rangle, |1\rangle\}$ be its canonical orthonormal basis, where $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The quantum computational counterpart of the *bit* - the basic information quantity of classical information theory - is the *quantum bit* (*qubit*), i.e. any *unit vector* $|\psi\rangle$ in $\mathbb{C}^2$. The general form of a qubit is:

$$|\psi\rangle = a_0 |0\rangle + a_1 |1\rangle,$$

where a, b are complex numbers s.t. $|a_0|^2 + |a_1|^2 = 1$. Qubits, therefore, correspond to pure states: in fact, as dictated by the *Born rule*,

- $|a_0|^2$ yields the probability of the information described by the pure state $|0\rangle$, which, from a logical viewpoint, corresponds to *falsity;*
- $|a_1|^2$ yields the probability of the information described by the pure state $|1\rangle$, corresponding to *truth.*

Therefore, $|0\rangle$ and $|1\rangle$ represent *maximal and certain* pieces of information, while a *superposition* $|\psi\rangle$ (in other words, a linear combination with nonzero coefficients of the basis vectors $|0\rangle$ and $|1\rangle$) corresponds to a *maximal* but *uncertain* piece of information.

So far, so good. However, what physical meaning can we attach to superposition states? A superposition $|\psi\rangle$ of the states $|0\rangle$ and $|1\rangle$ is a new state absolutely distinct from both $|0\rangle$ and $|1\rangle$; this typically holistic phenomenon is known as *the superposition principle* (2). For example, consider an idealized atom with a single electron and two energy levels: a *ground state* (identified with $|0\rangle$), which we suppose to be the current state of the electron, and an *excited state* (identified with $|1\rangle$). By shining a light pulse of half the duration as the one needed to perform a change of the energy level from $|0\rangle$ to $|1\rangle$, we can effect a "half-flip" between the two logical states. The ensuing state of the atom is neither $|0\rangle$ nor $|1\rangle$, but rather a superposition of both states. The electron is neither in the ground state, nor in the excited state, but "halfway in between".

Suppose, now, that we measure the energy of such an electron. The measurement process will not admit an uncertain result: the electron must be detected in either one of the two levels. The respective probabilities that the electron will be detected

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in the ground or in the excited level will be $|a_0|^2$ and $|a_1|^2$. That is, the electron has changed again its energy level since the measurement procedure "has forced" $|\psi\rangle$ to collapse into one of the two possible states. In some sense (see e.g. (10)) the measurement procedure did not produce any information about the way $|\psi\rangle$ was before the measurement, but caused an irreversible change of the initial state $|\psi\rangle$.

2.2   *Tensor spaces, factorized states, quregisters*

Suppose we have to deal with a physical system S composed by n component subsystems, say $S_1, ..., S_n$. Let $\mathcal{H}^{S_i}$ be the Hilbert spaces associated with S_i, for $1 \leq i \leq n$. The space $\mathcal{H}$ associated to S will be the *tensor product* $\mathcal{H}^{S_1} \otimes ... \otimes \mathcal{H}^{S_n}$ (see (28)) of the spaces associated with $S_1, ..., S_n$. If $S_i = S_j$ for every i, j, we resort to the notation $\overset{n}{\bigotimes} \mathcal{H}^{S_i}$ in place of $\underbrace{\mathcal{H}^{S_i} \otimes ... \otimes \mathcal{H}^{S_i}}_{n-\text{times}}$. Once again, the space $\mathcal{H}$ will be "something different" from the spaces $\mathcal{H}^{S_1}, ..., \mathcal{H}^{S_n}$. Given m vector spaces $\mathcal{H}^{S_1}, ..., \mathcal{H}^{S_m}$ and a state $|\psi\rangle \in \mathcal{H}^{S_1} \otimes ... \otimes \mathcal{H}^{Sm}$, we call $|\psi\rangle$ a *factorized state* iff $|\psi\rangle = |\psi_1\rangle \otimes ... \otimes |\psi_m\rangle$, for $|\psi_j\rangle \in \mathcal{H}^{Sj}$ and $1 \leq j \leq m$ (see e.g. (9)). In general, it is not the case that every vector in a tensor product space is amenable to factorization; *entangled states*, in fact, are noncontrastable states, i.e. there is no way to express them as tensor products of pure states of the component subsystems $S_1, ..., S_n$. [1]

As we have seen, qubits "live" in the space $\mathbb{C}^2$. *Quregisters* are the tensor product analogues of qubits: by quregister, in fact, we mean any unit vector in $\overset{n}{\bigotimes} \mathbb{C}^2$. By way of example, consider the space $\overset{3}{\bigotimes} \mathbb{C}^2$, whose canonical basis is

$$\{|000\rangle, |001\rangle, |010\rangle, |011\rangle, |100\rangle, |101\rangle, |110\rangle, |111\rangle\}.$$

A quregister will be a vector

$$|\phi\rangle = a_0 |000\rangle + a_1 |001\rangle + a_2 |010\rangle + a_3 |011\rangle + a_4 |100\rangle + a_5 |101\rangle + a_6 |110\rangle + a_7 |111\rangle,$$

where the a_i's are complex numbers s.t. $\sum_{i=0}^{7} |a_i|^2 = 1$.

We will call any factorized unit vector $|\phi\rangle = |x_1, ..., x_n\rangle$ of $\overset{n}{\bigotimes} \mathbb{C}^2$, where $x_1, ..., x_n$ are variables ranging over the set $\{0, 1\}$, an $n-configuration$[2]. It is not hard to see that one can identify each $n-$configuration with a natural number $i \in [0, 2^n - 1]$, for $i = 2^{n-1}x_1 + 2^{n-2}x_2 + ... + x_n$; intuitively, any $n-$configuration can be read as a natural number in its binary codification. In other words, one can concisely express a quregister $|\phi\rangle$ as

$$|\phi\rangle = \sum_{j=0}^{2^n-1} c_j \, || \, j \rangle\rangle,$$

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[1] For the sake of simplicity we henceforth denote $|\psi_1\rangle \otimes |\psi_2\rangle \otimes ... \otimes |\psi_m\rangle$ by $|\psi_1, \psi_2, ..., \psi_m\rangle$ or $|\psi_1\psi_2...\psi_m\rangle$.

[2] The set of all $n-$configurations $\mathcal{B}^{(n)} = \{|x_1, ..., x_n\rangle : x_i \in \{0, 1\}\}$ is an orthonormal basis for $\overset{n}{\bigotimes} \mathbb{C}^2$. We call $\mathcal{B}^{(n)}$ the *computational basis* of $\overset{n}{\bigotimes} \mathbb{C}^2$.

where c_j is a complex number, $|| j \rangle\rangle$ is the $n-$configuration corresponding to the number j, and $\sum_{j=0}^{2^n-1} |c_j|^2 = 1$.

Let $\mathfrak{R}\left(\overset{n}{\bigotimes}\mathbb{C}^2\right)$ be the set of all quregisters of $\overset{n}{\bigotimes}\mathbb{C}^2$. We denote by

$$\mathfrak{R} := \bigcup_{n=1}^{\infty}\left(\mathfrak{R}\left(\overset{n}{\bigotimes}\mathbb{C}^2\right)\right)$$

the set of all quregisters in $\mathbb{C}^2$ or in a tensor product of $\mathbb{C}^2$.

2.3   *The mathematical framework of qumixes*

Non-maximal pieces of information are matched, on a mathematical level, by *qumixes*, i.e. density operators[1] on $\mathbb{C}^2$ or on appropriate tensor products $\overset{n}{\bigotimes}\mathbb{C}^2$ of $\mathbb{C}^2$. Let us, first, introduce the appropriate mathematical framework.

Consider the following two sets of natural numbers:

$$C_1^{(n)} = \{i| \, || i \rangle\rangle = |x_1, ..., x_n\rangle \text{ and } x_n = 1\},$$
$$C_0^{(n)} = \{j| \, || j \rangle\rangle = |x_1, ..., x_n\rangle \text{ and } x_n = 0\}.$$

Let us focus on a generic quregister in $\overset{n}{\bigotimes}\mathbb{C}^2$:

$$|\phi\rangle = \sum_{k=0}^{2^n-1} a_k \, || k \rangle\rangle.$$

We can rewrite $|\phi\rangle$ as

$$|\phi\rangle = \sum_{i \in C_1^{(n)}} a_i \, || i \rangle\rangle + \sum_{j \in C_0^{(n)}} a_j \, || j \rangle\rangle.$$

Let $P_1^{(n)}$ and $P_0^{(n)}$ be the projection operators [2] onto the subspaces spanned by the sets $\left\{ || i \rangle\rangle \, | i \in C_1^{(n)} \right\}$ and $\left\{ || j \rangle\rangle \, | j \in C_0^{(n)} \right\}$, respectively. It is immediate to see that $P_1^{(n)} + P_0^{(n)} = I^{(n)}$, where $I^{(n)}$ is the identity operator of $\overset{n}{\bigotimes}\mathbb{C}^2$. It is clear that $P_1^{(n)}$ and $P_0^{(n)}$ are density operators iff $n = 1$ (if $n \neq 1$, we apply a normalization coefficient $k_n = \frac{1}{2^{n-1}}$ in such a way that $k_n P_1^{(n)}$ and $k_n P_0^{(n)}$ are density operators).

From an intuitive point of view, $P_1^{(n)}$ and $P_0^{(n)}$ can be regarded as the mathematical representatives of the *truth property* and the *falsity property*, respectively, in the space $\overset{n}{\bigotimes}\mathbb{C}^2$. Clearly, in $\mathbb{C}^2$, the projections $P_1^{(1)}$ and $P_0^{(1)}$ correspond, respectively, to the qubits $|1\rangle$ and $|0\rangle$.

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[1] A density operator is a positive, self-adjoint, trace class (linear) operator with trace 1.
[2] A projection operator is a self-adjoint operator ρ s.t. $\rho^2 = \rho$.

Let $\mathfrak{D}\left(\overset{n}{\bigotimes}\mathbb{C}^2\right)$ be the set of all density operators on $\overset{n}{\bigotimes}\mathbb{C}^2$. We denote by

$$\mathfrak{D} := \bigcup_{n=1}^{\infty}\left(\mathfrak{D}\left(\overset{n}{\bigotimes}\mathbb{C}^2\right)\right)$$

the set of all density operators in $\mathbb{C}^2$ or in a tensor product of $\mathbb{C}^2$.

This set is a convenient representation of the set of all qumixes. Any quregister can be regarded as a limiting case of a qumix: a quregister is a density operator which is also a projection operator.

If $\rho \in \mathfrak{D}\left(\overset{n}{\bigotimes}\mathbb{C}^2\right)$ is a qumix, its *probability* $\mathrm{p}(\rho)$ is $\mathrm{tr}\left(P_1^{(n)}\rho\right)$, where tr is the trace functional. Intuitively, $\mathrm{p}(\rho)$ represents the probability that the information stored by the qumix ρ is true. When ρ corresponds to the qubit

$$|\phi\rangle = a_0\,|0\rangle + a_1\,|1\rangle,$$

it turns out that $\mathrm{p}(\rho) = |a_1|^2$.

2.4   *Quantum gates*

Let us now return to quantum information units - qubits, quregisters and qumixes. Similarly to the classical case, we can introduce and study the behavior of a number of *quantum logical gates* operating on them. These gates are mathematically represented by *unitary operators* on the appropriate Hilbert spaces ((4), (10), (11)). The unitarity property is required to guarantee *reversibility*, a fact marking a fundamental difference with usual classical computation.

In fact, if we consider the classical And truth table, it is immediate to see that it represents a typical many-to-one irreversible transformation, for it is impossible in general to retrieve the input values from a given output:

$$(0,0) \to 0$$
$$(0,1) \to 0$$
$$(1,0) \to 0$$
$$(1,1) \to 1.$$

Instead, the quantum And presupposes the introduction of a special unitary operator (the so-called *Petri-Toffoli gate* or simply the *Toffoli gate*). For any $m, n \geq 1$, the *Petri-Toffoli* gate is the unitary operator $T^{(m,n,1)}$ such that, for every element $|x_1, ..., x_m\rangle \otimes |y_1, ..., y_n\rangle \otimes |z\rangle$ of the computational basis $\mathcal{B}^{(m+n+1)}$ (shortened as $|x\rangle \otimes |y\rangle \otimes |z\rangle$),

$$T^{(m,n,1)}(|x\rangle \otimes |y\rangle \otimes |z\rangle) = |x\rangle \otimes |y\rangle \otimes \left|x_m y_n \widehat{+} z\right\rangle,$$

where $\widehat{+}$ represents the sum modulo 2. For instance, $T^{(1,1,1)}$ transforms any factorized vector $|x\rangle \otimes |y\rangle \otimes |z\rangle$ into the vector obtained by leaving the first two factors (referred to as the *control bits*) unchanged, while replacing $|z\rangle$ (the *target bit*) by $\left|xy\widehat{+}z\right\rangle$. This yields the following "table":

$$|000\rangle \rightarrow |000\rangle$$
$$|001\rangle \rightarrow |001\rangle$$
$$|010\rangle \rightarrow |010\rangle$$
$$|011\rangle \rightarrow |011\rangle$$
$$|100\rangle \rightarrow |100\rangle$$
$$|101\rangle \rightarrow |101\rangle$$
$$|110\rangle \rightarrow |111\rangle$$
$$|111\rangle \rightarrow |110\rangle .$$

$T^{(1,1,1)}$ behaves like the identity matrix on the first six basis elements, while interchanging the last two basis elements. The matrix representation of $T^{(1,1,1)}$ relative to the computational basis is the following:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The operator $T^{(m,n,1)}$ affords a convenient notion of *conjunction*. This conjunction (And) is characterized as a function whose arguments are vectors $|\psi\rangle$, $|\varphi\rangle$ of $\bigotimes^m \mathbb{C}^2$ and $\bigotimes^n \mathbb{C}^2$, respectively and whose values are vectors of the product space $\bigotimes^{m+n+1} \mathbb{C}^2$. If $|\psi\rangle \in \bigotimes^m \mathbb{C}^2$ and $|\varphi\rangle \in \bigotimes^n \mathbb{C}^2$, we define

$$\text{And}(|\psi\rangle, |\varphi\rangle) = T^{(m,n,1)}(|\psi\rangle \otimes |\varphi\rangle \otimes |0\rangle).$$

In the above definition, $|0\rangle$ represents an *ancilla* which increases the dimension of the space, but renders the operator reversible. For $|\psi\rangle = |0\rangle$ and $|\varphi\rangle = |1\rangle$ we obtain the following typically reversible (one-to-one) table for And:

$$|00\rangle \rightarrow |000\rangle$$
$$|01\rangle \rightarrow |010\rangle$$
$$|10\rangle \rightarrow |100\rangle$$
$$|11\rangle \rightarrow |111\rangle .$$

2.5   *Semiclassical and genuinely quantum gates*

A gate A is *semiclassical* if its outputs cannot be superposition states whenever its inputs are not superposition states. The label "semiclassical" is being used since such gates behave just like their respective boolean counterparts whenever they

are applied to non-superposition inputs; nevertheless, unlike classical gates, they can also be applied to superposition states.

A typical example, beside the quantum And, is the quantum Not. For any $n \geq 1$, the *negation* on $\overset{n}{\bigotimes} \mathbb{C}^2$ is the unitary operator $\text{Not}^{(n)}$ such that, for every element $|x_1, ..., x_n\rangle$ of the computational basis $\mathcal{B}^{(n)}$,

$$\text{Not}^{(n)}(|x_1, ..., x_n\rangle) = |x_1, ..., x_{n-1}\rangle \otimes |1 - x_n\rangle.$$

We have that:

$$\text{Not}^{(n)} = \begin{cases} \sigma_x & \text{if } n = 1; \\ I^{(n-1)} \otimes \sigma_x, & \text{otherwise,} \end{cases}$$

where $\sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the "first" Pauli matrix.

A gate is *genuinely quantum* if it is not semiclassical. A remarkable case in point is the *square root of the negation* $\sqrt{\text{Not}}$. For any $n \geq 1$, the square root of the negation on $\overset{n}{\bigotimes} \mathbb{C}^2$ is the unitary operator $\sqrt{\text{Not}}^{(n)}$ such that, for every element $|x_1, ..., x_n\rangle$ of the computational basis $\mathcal{B}^{(n)}$,

$$\sqrt{\text{Not}}^{(n)}(|x_1, ..., x_n\rangle) = |x_1, ..., x_{n-1}\rangle \otimes \frac{1}{2}\left((1+i)|x_n\rangle + (1-i)|1 - x_n\rangle\right).$$

The basic property of $\sqrt{\text{Not}}^{(n)}$ is the following: for any $|\psi\rangle \in \overset{n}{\bigotimes} \mathbb{C}^2$,

$$\sqrt{\text{Not}}^{(n)}\left(\sqrt{\text{Not}}^{(n)}(|\psi\rangle)\right) = \text{Not}^{(n)}(|\psi\rangle).$$

From a logical point of view, therefore, the square root of the negation can be regarded as a kind of "tentative partial negation" that transforms precise pieces of information into maximally uncertain ones. For, we have

$$\text{p}(\sqrt{\text{Not}}^{(1)}(|0\rangle)) = \frac{1}{2} = \text{p}(\sqrt{\text{Not}}^{(1)}(|1\rangle)).$$

Our quantum gate $\sqrt{\text{Not}}$ has no boolean counterpart ((10)). The "half-flip" mentioned in the idealized atom example of Section 2.1 is a natural physical model for this gate.

Let us proceed with another useful genuinely quantum gate. For any $n \geq 1$, the *square root of the identity* on $\overset{n}{\bigotimes} \mathbb{C}^2$ is the linear operator $\sqrt{\text{I}}^{(n)}$ such that for every element $|x_1, ..., x_n\rangle$ of the computational basis $\mathcal{B}^{(n)}$:

$$\sqrt{\text{I}}^{(n)}(|x_1, ..., x_n\rangle) = |x_1, ..., x_{n-1}\rangle \otimes \frac{1}{\sqrt{2}}\left((-1)^{x_n}|x_n\rangle + |1 - x_n\rangle\right).$$

We have that

$$\sqrt{\text{I}}^{(n)} = \begin{cases} H & \text{if } n = 1; \\ I^{n-1} \otimes H, & \text{otherwise,} \end{cases}$$

where H is the *Hadamard* matrix:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

The basic property of $\sqrt{I}^{(n)}$ is the following: for any $|\psi\rangle \in \overset{n}{\bigotimes} \mathbb{C}^2$:

$$\sqrt{I}^{(n)} \left(\sqrt{I}^{(n)}(|\psi\rangle) \right) = |\psi\rangle.$$

Logically speaking, thus, $\sqrt{I}^{(n)}$ can be seen as a "tentative partial assertion".

2.6 *Genuinely entangled gates*

Within the set of genuinely quantum gates, we can isolate a notable subset: the *computationally locally entangled gates.* Let us consider a special case first. A unitary operator U on $\overset{n}{\bigotimes} \mathbb{C}^2$ is *computationally entangled* if there exists a vector $|x_1, \ldots, x_n\rangle$ of the computational basis $\mathcal{B}^{(n)}$ such that $U(|x_1, \ldots, x_n\rangle)$ is an entangled state[1]. Now, upon inductively defining, for any two unitary operators $U, V \in \overset{m}{\bigotimes} \mathbb{C}^2$,

$$U \otimes_0 V = V;$$
$$U \otimes_{n+1} V = U \otimes (U \otimes_n V),$$

we say that U is *computationally locally entangled* iff there exists $m \geq 0$ and a computationally entangled gate W such that $U = I \otimes_m W$. Clearly, any computationally entangled gate is computationally locally entangled, for it suffices to fix $m = 0$.

A relevant example is as follows. For any $n \geq 1$, the *square root of swap* on $\overset{n}{\bigotimes} \mathbb{C}^2$ is the unitary operator $\sqrt{\mathrm{Swp}}^{(n)}$ such that, for every element $|x_1, ..., x_n\rangle$ of the computational basis $\mathcal{B}^{(n)}$:

$$\sqrt{\mathrm{Swp}}^{(n)}(|x_1, ..., x_n\rangle) = \begin{cases} \frac{1}{2} \left((1+i)\, |x_{n-1} x_n\rangle + (1-i)\, |x_n x_{n-1}\rangle \right), & \text{if } n = 2; \\ |x_1, ..., x_{n-2}\rangle \otimes \frac{1}{2} \left((1+i)\, |x_{n-1} x_n\rangle + (1-i)\, |x_n x_{n-1}\rangle \right), & \text{if } n > 2. \end{cases}$$

Its name stems from the basic property of $\sqrt{\mathrm{Swp}}^{(n)}$: by applying it twice to a given quregister, the target bits are "swapped". The matrix representation of $\sqrt{\mathrm{Swp}}^{(2)}$ is the following:

$$\sqrt{\mathrm{Swp}}^{(2)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1+i}{2} & \frac{1-i}{2} & 0 \\ 0 & \frac{1-i}{2} & \frac{1+i}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

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[1]Remark that there exist unitary gates which may have entangled states as outputs, yet fail to be computationally entangled. A case in point is the XOR gate, which yields an entanglement only when applied to superposition states (cf. (25)).

The quantum gate $\sqrt{\mathrm{Swp}}^{(2)}$ is computationally entangled: if we apply it to the basis elements $|10\rangle$ and $|01\rangle$ we get entangled states as outputs. On the other hand, $\sqrt{\mathrm{Swp}}^{(3)}$ is computationally locally entangled in that $\sqrt{\mathrm{Swp}}^{(3)} = I \otimes_1 \sqrt{\mathrm{Swp}}^{(2)}$. If we apply $\sqrt{\mathrm{Swp}}^{(3)}$ to an element $|x_1 x_2 x_3\rangle$ of the computational basis of $\overset{3}{\bigotimes} \mathbb{C}^2$, our output is $|x_1\rangle \otimes \frac{1}{2}\left((1+i)|x_2 x_3\rangle + (1-i)|x_3 x_2\rangle\right)$. It is essential to remark that, although $\frac{1}{2}\left((1+i)|x_2 x_3\rangle + (1-i)|x_3 x_2\rangle\right)$ is an entangled state, the whole output is a factorized state with factors in $\mathbb{C}^2$ and in $\mathbb{C}^4$.

3.   Quantum computational logics

Interestingly enough, qumixes and quregisters are connected with the real closed unit interval $[0,1]$. In fact, given a real number $\lambda \in [0,1]$ and an $n \in \mathbb{N}^+$, and recalling that $k_n = \frac{1}{2^{n-1}}$, we can define an n-quregister $|\psi\rangle_\lambda$ and a qumix $\rho_\lambda^{(n)}$ in the following way:

- $|\psi\rangle_\lambda = \begin{cases} \sqrt{1-\lambda}\,|0\rangle + \sqrt{\lambda}\,|1\rangle, & \text{if } n = 1; \\ \sqrt{(1-\lambda)\,k_n} \sum_{j=0}^{2^{n-1}-1} \|\,j\rangle\rangle\,|0\rangle + \sqrt{\lambda k_n} \sum_{j=0}^{2^{n-1}-1} \|\,j\rangle\rangle\,|1\rangle, & \text{if } n > 1. \end{cases}$
- $\rho_\lambda^{(n)} = (1-\lambda)\,k_n P_0^{(n)} + \lambda k_n P_1^{(n)}$.

The quregister $|\psi\rangle_\lambda \in \mathfrak{R}\left(\overset{n}{\bigotimes} \mathbb{C}^2\right)$ can be interpreted as the maximal information that might correspond to the truth with probability λ, while $\rho_\lambda^{(n)} \in \mathfrak{D}\left(\overset{n}{\bigotimes} \mathbb{C}^2\right)$ represents a "mixture" of information pieces that might correspond to the truth with probability λ. Some relevant properties of $|\psi\rangle_\lambda$ and $\rho_\lambda^{(n)}$ are summarized in the following Lemmas:

Lemma 3.1:   *(12)*

(1) $\forall n \in \mathbb{N}^+ \forall \lambda \in [0,1]$*:* $\mathrm{p}\left(|\psi\rangle_\lambda\right) = \lambda$*;*

(2) $\mathrm{p}\left(\sqrt{\mathrm{Not}}\,|\psi\rangle_\lambda\right) = \frac{1}{2}$*;*

(3) $\mathrm{p}\left(\sqrt{\mathrm{I}}\,|\psi\rangle_\lambda\right) = \frac{1}{2} - \sqrt{(1-\lambda)\,\lambda}$*.*

We now settle on a notational convention whose aim is to permit an extension to qumixes of the gates we defined above in the framework of quregisters.

Notation 3.2:   *For any qumixes* $\sigma \in \mathfrak{D}\left(\overset{m}{\bigotimes} \mathbb{C}^2\right)$ *and* $\tau \in \mathfrak{D}\left(\overset{m}{\bigotimes} \mathbb{C}^2\right)$,

$$\sqrt{\mathrm{NOT}}^{(m)}\sigma = \sqrt{\mathrm{Not}}^{(m)}\sigma\sqrt{\mathrm{Not}}^{(m)*};$$

$$\sqrt{\mathbf{I}}^{(m)}\sigma = \sqrt{\mathrm{I}}^{(m)}\sigma\sqrt{\mathrm{I}}^{(m)*};$$

$$\mathrm{NOT}^{(m)}\sigma = \mathrm{Not}^{(m)}\sigma\mathrm{Not}^{(m)*};$$

$$\mathrm{AND}^{(m,n,1)}(\sigma, \tau) = \mathbf{T}^{(m,n,1)}\left(\sigma, \tau, P_0^{(1)}\right)$$

$$:= T^{(m,n,1)}\left(\sigma \otimes \tau \otimes P_0^{(1)}\right) T^{(m,n,1)},$$

where a suffixed * *denotes the adjoint operator.*

Using this notation, Lemma 3.1 carries over to qumixes as follows:

Lemma 3.3: *(12)*

(1) $\forall n \in \mathbb{N}^+ \forall \lambda \in [0,1]$*:* $\mathrm{p}\left(\rho_\lambda^{(\mathrm{n})}\right) = \lambda$*;*

(2) $\mathrm{p}\left(\sqrt{\mathrm{NOT}}\rho_\lambda^{(\mathrm{n})}\right) = \frac{1}{2}$*;*

(3) $\mathrm{p}\left(\sqrt{\mathbf{I}}\rho_\lambda^{(\mathrm{n})}\right) = \frac{1}{2}$*.*

Let us take stock. So far we have been concerned with the introduction of quantum information units - in the most general case, qumixes - and of appropriate operations on such quantum logical gates. In the present section, we show how to concoct a few *logics* out of these ingredients. We want to select the set $\mathfrak{D}$ of all qumixes as the common universe of a series of first order structures which share a fixed set of operations corresponding to quantum logical gates, but differ from one another with respect to their unique relation (which in all these cases is a preorder relation). These relations are defined hereafter for $\rho, \sigma \in \mathfrak{D}$.

Definition 3.4: (12) Weak preorder

$$\rho \preceq_w \sigma \ \ \text{iff} \ \ \mathrm{p}\left(\rho\right) \leq \mathrm{p}\left(\sigma\right).$$

Definition 3.5: (12) Strong preorder

$$\rho \preceq_s \sigma \ \ \text{iff} \ \ \mathrm{p}\left(\rho\right) \leq \mathrm{p}\left(\sigma\right) \ \text{and} \ \mathrm{p}\left(\sqrt{\mathrm{NOT}}\rho\right) \leq \mathrm{p}\left(\sqrt{\mathrm{NOT}}\sigma\right).$$

Definition 3.6: (12) Super-strong preorder

$$\rho \preceq_{ss} \sigma \ \ \text{iff} \ \mathrm{p}\left(\rho\right) \leq \mathrm{p}\left(\sigma\right) \ \text{and} \ \mathrm{p}\left(\sqrt{\mathrm{NOT}}\rho\right) \leq \mathrm{p}\left(\sqrt{\mathrm{NOT}}\sigma\right) \ \text{and} \ \mathrm{p}\left(\sqrt{\mathbf{I}}\rho\right) \leq \mathrm{p}\left(\sqrt{\mathbf{I}}\sigma\right).$$

As the names suggest, the superstrong preorder is stronger than the strong, which is in turn stronger than the weak:

$$\text{if} \ \ \rho \preceq_{ss} \sigma \ \ \text{then} \ \ \rho \preceq_s \sigma;$$
$$\text{if} \ \ \rho \preceq_s \sigma \ \ \text{then} \ \ \rho \preceq_w \sigma.$$

We can finally define:

(1) The *standard reversible weak quantum computational structure* (briefly, *WQC*):

$$\left(\mathfrak{D}, \preceq_w, \mathrm{AND}, \mathrm{NOT}, \sqrt{\mathrm{NOT}}, \sqrt{\mathbf{I}}, P_0^{(1)}, P_1^{(1)}, \rho_{\frac{1}{2}}^{(1)}\right)$$

(2) The *standard reversible strong quantum computational structure* (briefly, *SQC*):

$$\left(\mathfrak{D}, \preceq_s, \mathrm{AND}, \mathrm{NOT}, \sqrt{\mathrm{NOT}}, \sqrt{\mathbf{I}}, P_0^{(1)}, P_1^{(1)}, \rho_{\frac{1}{2}}^{(1)}\right)$$

(3) The *standard reversible superstrong quantum computational structure* (briefly, *SSQC*):

$$\left(\mathfrak{D}, \preceq_{ss}, \mathrm{AND}, \mathrm{NOT}, \sqrt{\mathrm{NOT}}, \sqrt{\mathbf{I}}, P_0^{(1)}, P_1^{(1)}, \rho_{\frac{1}{2}}^{(1)}\right)$$

Intuitively, $P_0^{(1)}, P_1^{(1)}, \rho_{\frac{1}{2}}^{(1)}$ represent special pieces of information, that are *false*, *true*, *indeterminate*, respectively. In what follows, for the sake of notational clarity, we will omit the superscripts whenever no danger of confusion is impending.

Up to now, we presented only reversible gates. Nevertheless, it is also possible to endow $\mathfrak{D}$ with a kind of *irreversible conjunction*:

Definition 3.7: The irreversible conjunction
For any qumixes $\sigma \in \mathfrak{D}\left(\overset{m}{\bigotimes}\mathbb{C}^2\right)$ and $\tau \in \mathfrak{D}\left(\overset{n}{\bigotimes}\mathbb{C}^2\right)$,

$$\mathrm{IAND}\,(\sigma, \tau) := \rho_{\mathrm{p}(\sigma)\mathrm{p}(\tau)}^{(1)}.$$

It should be noted that the output of an irreversible conjunction is a qumix of $\mathbb{C}^2$. Remark also that $\mathrm{p}(\sigma)\,\mathrm{p}(\tau)$ is a real number in the closed unit interval $[0, 1]$. Whenever it is $\neq 1$, there exist infinitely many values of $\mathrm{p}(\sigma)$ and $\mathrm{p}(\tau)$ yielding that very product. In other words, only if $\mathrm{p}(\sigma) = \mathrm{p}(\tau) = 1$ no infinite number of factorizations is allowed. This means that $\rho_{\mathrm{p}(\sigma)\mathrm{p}(\tau)}^{(1)}$ will be the density operator $P_1^{(1)}$ associated to $|1\rangle$ only if $\sigma = \tau = P_1^{(1)}$.

Besides the IAND, another example of irreversible transformation is represented by a *Łukasiewicz-like disjunction*:

Definition 3.8: The Łukasiewicz disjunction.
Let $\sigma \in \mathfrak{D}\left(\overset{m}{\bigotimes}\mathbb{C}^2\right)$ and $\tau \in \mathfrak{D}\left(\overset{n}{\bigotimes}\mathbb{C}^2\right)$:

$$\sigma \oplus \tau := \rho_{\mathrm{p}(\sigma)\oplus\mathrm{p}(\tau)}^{(1)},$$

where $\oplus$ is the Łukasiewicz "truncated sum", i.e. $\min(x+y, 1)$, for $x, y \in [0, 1]$ (see e.g. (7)).

As one can easily see, the unique "reversible" application of the Łukasiewicz disjunction turns out to be when $\sigma = P_0^{(\mathrm{n})}$ and $\tau = P_0^{(\mathrm{m})}$.

3.1 *Reversible and irreversible models*

Let us consider a *minimal quantum computational language* $\mathcal{L}$ containing a designated atomic sentence $\mathbf{f}$, whose intuitive interpretation is the false. The language $\mathcal{L}$ contains three unary connectives - a *negation* ($\neg$), a *square root of the negation* ($\sqrt{\neg}$), a *square root of the identity* ($\sqrt{id}$) - and one binary connective - a conjunction ($\wedge$). We denote by $Var^{\mathcal{L}}$ and $Form^{\mathcal{L}}$, respectively, the sets of propositional variables and formulae of $\mathcal{L}$. As usual, we define $\alpha \vee \beta = \neg(\neg\alpha \wedge \neg\beta)$ and the truth constant $\mathbf{t}$ as $\neg\mathbf{f}$. We want to interpret any sentence α of this language by means of an appropriate qumix, depending on the logical form of α. We are now ready to introduce the definition of *reversible quantum computational model* (*RQC-model*, for short):

Definition 3.9: A RQC-model of $\mathcal{L}$ is a function $\mathrm{Qum} : Var^{\mathcal{L}} \to \mathfrak{D}$ which is inductively extended to all of $Form^{\mathcal{L}}$ as follows:

$$\mathrm{Qum}(\alpha) = \begin{cases} P_0 \text{ if } \alpha = \mathbf{f}; \\ \mathrm{NOT}\,(\mathrm{Qum}(\beta)) \text{ if } \alpha = \neg\beta; \\ \sqrt{\mathrm{NOT}}\,(\mathrm{Qum}(\beta)) \text{ if } \alpha = \sqrt{\neg}\beta; \\ \sqrt{\mathbf{I}}\,(\mathrm{Qum}(\beta)) \text{ if } \alpha = \sqrt{id}\beta; \\ \mathbf{T}\,(\mathrm{Qum}(\beta), \mathrm{Qum}(\gamma), P_0) \text{ if } \alpha = \beta \wedge \gamma. \end{cases}$$

Let us stress an important *quasi-intensional feature* of RQC-models: the meaning $\mathrm{Qum}(\alpha)$ of a sentence α depends on the logical form of α - the more complex the sentence, the higher the dimension of the space where $\mathrm{Qum}(\alpha)$ "lives".

According to which preorder relation - $\preceq_w$, $\preceq_s$ or $\preceq_{ss}$ - we choose to select, three corresponding notions of logical consequence[1] and logical truth arise:

Definition 3.10: β is, respectively, a *weak, strong,* or *super-strong consequence* in Qum of α ($\alpha \models^c_{\mathrm{Qum}} \beta$) iff $\mathrm{Qum}(\alpha) \preceq_c \mathrm{Qum}(\beta)$ (where c is, respectively, w, s or ss).

Definition 3.11: β is, respectively, *weakly, strongly,* or *super-strongly true* in Qum iff $\mathbf{t} \models^c_{\mathrm{Qum}} \alpha$ (where c is, respectively, w, s or ss).

Definition 3.12: β is, respectively, a *weak, strong,* or *super-strong logical consequence* of α ($\alpha \models^c_{RQC} \beta$) iff for any RQC-model Qum, $\alpha \models^c_{\mathrm{Qum}} \beta$ (where c is, respectively, w, s or ss).

Definition 3.13: β is, respectively, a *weak, strong,* or *super-strong logical truth* iff, for any RQC-model Qum, α is weakly (resp. strongly, super-strongly) true in Qum.

Definition 3.12 semantically introduces, *de facto*, three quantum computational logics which will be respectively denoted by $\mathbf{QCL}^w$, $\mathbf{QCL}^s$ and $\mathbf{QCL}^{ss}$. We have, of course, that $\mathbf{QCL}^{ss} \subset \mathbf{QCL}^s \subset \mathbf{QCL}^w$. The label $\mathbf{QCL}$ will ambiguously refer to any one between $\mathbf{QCL}^w$, $\mathbf{QCL}^s$, $\mathbf{QCL}^{ss}$.

A definition of *irreversible quantum computational model* (*IQC-model*, for short) now follows. Remark that we trade the irreversible gate IAND for Petri-Toffoli, to the effect that the dimension of the Hilbert space never increases, and all our formulae can be assigned a meaning in $\mathbb{C}^2$.

Definition 3.14: An IQC-model of $\mathcal{L}$ is a function $\mathrm{Qum}^{\mathbb{C}^2} : Var^{\mathcal{L}} \to \mathfrak{D}\left(\mathbb{C}^2\right)$ which is inductively extended to all of $Form^{\mathcal{L}}$ as follows:

$$\mathrm{Qum}^{\mathbb{C}^2}(\alpha) = \begin{cases} P_0 \text{ if } \alpha = \mathbf{f}; \\ \mathrm{NOT}\left(\mathrm{Qum}^{\mathbb{C}^2}(\beta)\right) \text{ if } \alpha = \neg\beta; \\ \sqrt{\mathrm{NOT}}\left(\mathrm{Qum}^{\mathbb{C}^2}(\beta)\right) \text{ if } \alpha = \sqrt{\neg}\beta; \\ \sqrt{\mathbf{I}}\left(\mathrm{Qum}^{\mathbb{C}^2}(\beta)\right) \text{ if } \alpha = \sqrt{id}\beta; \\ \mathrm{IAND}\left(\mathrm{Qum}^{\mathbb{C}^2}(\beta), \mathrm{Qum}^{\mathbb{C}^2}(\gamma)\right) \text{ if } \alpha = \beta \wedge \gamma. \end{cases}$$

---

[1] Remark that the meaning herein attached to the expression "logical consequence" only partially overlaps with the standard Tarskian notion of logical consequence relation adopted in contemporary abstract algebraic logic. To cite only the most striking difference, the present relation may or may not hold between single formulae, whereas Tarski's may or may not hold between a set of formulae and a single formula.

The notions of (weak, strong and super-strong) consequence, truth, logical consequence, and logical truth are defined, *mutatis mutandis*, as for the RQC-model case. In particular, we write $\alpha \models_{IQC}^c \beta$ whenever, for any IQC-model $\mathrm{Qum}^{\mathbb{C}^2}$, $\alpha \models_{\mathrm{Qum}^{\mathbb{C}^2}}^c \beta$. The label **IQCL** will ambiguously refer to any one between **IQCL**^{w}, **IQCL**^{s}, **IQCL**^{ss}.

The following theorem represents a crucial point in the study of these logics: each **QCL** and its irreversible match **IQCL** are one and the same logic.

Theorem 3.15: *(12) For $c \in \{w, s, ss\}$, $\alpha \models_{IQC}^c \beta$ iff $\alpha \models_{RQC}^c \beta$.*

Let us now investigate the role played by density operators in characterizing **QCL**. First of all, we introduce the notion of *reversible qubit model* (*RQB-model*, for short), where the meaning of any sentence α is given by a qubit.

Definition 3.16: A RQB-model of $\mathcal{L}$ is a function $\mathrm{Qub} : Var^{\mathcal{L}} \to \mathfrak{R}$ which is inductively extended to all of $Form^{\mathcal{L}}$ as follows:

$$\mathrm{Qub}(\alpha) = \begin{cases} |0\rangle \text{ if } \alpha = \mathbf{f}; \\ \mathrm{Not}\,(\mathrm{Qub}(\beta)) \text{ if } \alpha = \neg\beta; \\ \sqrt{\mathrm{Not}}\,(\mathrm{Qub}(\beta)) \text{ if } \alpha = \sqrt{\neg}\beta; \\ \sqrt{\mathrm{I}}\,(\mathrm{Qub}(\beta)) \text{ if } \alpha = \sqrt{id}\beta; \\ T\,(\mathrm{Qub}(\beta), \mathrm{Qub}(\gamma), |0\rangle) \text{ if } \alpha = \beta \wedge \gamma. \end{cases}$$

Again, the definitions of (weak, strong, super-strong) consequence, truth, logical consequence and logical truth are defined as for the RQC-model case. We will write $\alpha \models_{RQB}^c \beta$ (where $c \in \{w, s, ss\}$) when β is a (weak, strong, superstrong) logical consequence of α in the qubit semantics.

According to the next result, the weak and strong logical consequence relations over $\mathbb{C}^2$ remain unaltered upon moving to higher dimension spaces. From a purely logical point of view, "nothing is lost" if we restrict ourselves to $\mathbb{C}^2$.

Theorem 3.17: *(12) For $c \in \{w, s\}$, $\alpha \models_{RQC}^c \beta$ iff $\alpha \models_{RQB}^c \beta$.*

Theorem 3.17 cannot be extended to the case of **QCL**^{ss}: one can prove that, for any choice of a proper mixture $\rho \in \mathfrak{D}\left(\mathbb{C}^2\right)$, there exists no qubit $|\psi\rangle$ s.t. $\mathrm{p}\left(|\psi\rangle\right) = \mathrm{p}\left(\rho\right)$, $\mathrm{p}\left(\sqrt{\mathrm{Not}}\,|\psi\rangle\right) = \mathrm{p}\left(\sqrt{\mathrm{NOT}}\rho\right)$, and $\mathrm{p}\left(\sqrt{\mathrm{I}}\,|\psi\rangle\right) = \mathrm{p}\left(\sqrt{\mathbf{I}}\rho\right)$ - where it is the third equality that must be blamed for the failure ((12)).

3.2   *A geometrical insight*

As we have seen in Theorem 3.17, it is unnecessary - from a logical viewpoint - to consider information quantities in Hilbert spaces other than $\mathbb{C}^2$: the algebra whose universe is the set of all qumixes of $\mathbb{C}^2$ and whose operations correspond to appropriate extensions of the quantum logical gates generates the same logical consequence relation as the algebra over the set of all qumixes of arbitrary n-fold tensor products of $\mathbb{C}^2$.

In view of this result, let us provide a useful and intuitive representation of a qubit in its "living space" $\mathbb{C}^2$. Any generic qubit $|\psi\rangle = a_0 |0\rangle + a_1 |1\rangle$ can also be rewritten as

$$|\psi\rangle = e^{i\xi}\left(\cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle\right),$$

where θ, ϕ and ξ are real numbers. For the sake of simplicity, one can equivalently

write $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$, since the global phase $e^{i\xi}$ has no observable effects. As the values of θ and ϕ vary, we obtain all the unit vectors of $\mathbb{C}^2$ by picking one by one all the points on the surface of the unit three-dimensional closed sphere, also called the *Bloch-Poincaré sphere* $\mathbf{D}^3$. We have, thus, the following representation of a generic qubit $|\psi\rangle$ based on the spherical coordinate system $x = \cos\varphi\sin\theta$, $y = \sin\varphi\sin\theta$, $z = \cos\theta$. Notice that the qubits $|0\rangle$, $|1\rangle$ correspond to the North and to the South poles, respectively. As the value of θ increases within

Figure 1. The Bloch-Poincaré sphere $\mathbf{D}^3$

$[2n\pi, (2n+1)\pi]$, $n \in \mathbb{N}$, there is a corresponding increase of the probability that the information stocked in $|\psi\rangle$ is true.

In addition, let us recall that density operators of $\mathbb{C}^2$ are amenable to the well-known representation via the Pauli matrices:

$$\frac{1}{2}\left(\mathbb{I} + r_1\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + r_2\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} + r_3\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right),$$

where $\mathbb{I}$ is the identity 2×2 matrix and r_1, r_2, r_3 are real numbers such that $r_1^2 + r_2^2 + r_3^2 \leq 1$. Therefore, density operators are in one-one correspondence with the (inner or surface) points of the Bloch sphere. Clearly, if a density operator ρ is such that $r_1^2 + r_2^2 + r_3^2 = 1$, then ρ is a projection operator on $\mathbb{C}^2$.

This is not, however, the whole story. If we represent each density operator ρ as a triple $\langle r_1, r_2, r_3 \rangle$, as shown in the previous paragraph, the third element of the triple determines the probability of ρ, while the second element of the triple determine the probability of $\sqrt{\mathrm{NOT}}\rho$. In fact, an easy calculation shows that

$$\mathrm{p}(\rho) = \frac{1 - r_3}{2}, \quad \mathrm{p}\left(\sqrt{\mathrm{NOT}}\rho\right) = \frac{1 - r_2}{2}.$$

It follows that, if we are concerned only with the probability of ρ and with the probability of $\sqrt{\mathrm{NOT}}\rho$, we can shift down by one dimension: the triple $\langle r_1, r_2, r_3 \rangle$ shrinks to the pair $\langle a, b \rangle$, where a represents the probability of ρ and b represents the probability of $\sqrt{\mathrm{NOT}}\rho$. Clearly, the elements a, b must satisfy the condition that $a^2 + b^2 \leq 1$; that is, they must belong to the closed disc $\mathbf{D}^2$. To make computations easier, however, it is more convenient to transpose the disc to the first quadrant, scaling it down by one half: after such a move, qumixes are represented (modulo a neglection of the first component) by points of the closed disc with center $\langle\frac{1}{2}, \frac{1}{2}\rangle$ and radius $\frac{1}{2}$ - which correspond to the subset $\left\{\langle a, b \rangle \in \mathbb{R} \times \mathbb{R} : (1 - 2a)^2 + (1 - 2b)^2 \leq 1\right\}$ of the set of all *complex numbers*. In this way, quantum logical gates are transformed into operations on such a set of complex numbers, and we obtain some *standard algebras* over the complex numbers, sharing the same universe but having different signatures according to the set of logical gates under examination ((5), (12)).

In the next two sections, we will first show how to equip the above mentioned set of complex numbers with the Łukasiewicz-like OR of Definition 3.8 and the NOT gate, and then expand this structure by an operation corresponding to the $\sqrt{\mathrm{NOT}}$ gate.

4.   Quasi-MV algebras

In the previous section, we remarked that the set of all qumixes of $\mathbb{C}^2$ is in bijective correspondence with a subset of the unit complex interval $[\langle 0, 0\rangle, \langle 1, 1\rangle]$, i.e. with the lattice ordered set

$$D = \{\langle a, b\rangle : a, b \in \mathbb{R} \text{ and } (1 - 2a)^2 + (1 - 2b)^2 \leq 1\}.$$

Suppose we endow this set with operations corresponding (via the conventions and simplifications already mentioned in the previous section) to the Łukasiewicz-like OR of Definition 3.8 and the NOT gate:

$$\langle a, b\rangle \oplus^{\mathbf{D}} \langle c, d\rangle = \left\langle \min(1, a + c), \frac{1}{2}\right\rangle;$$

$$\langle a, b\rangle'^{\mathbf{D}} = \langle 1 - a, 1 - b\rangle,$$

and we select the designated elements $0^{\mathbf{D}} = \langle 0, \frac{1}{2}\rangle$ and $1^{\mathbf{D}} = \langle 1, \frac{1}{2}\rangle$. What we get is an algebra $\mathbf{D}$ of type $\langle 2, 1, 0, 0\rangle$, i.e. in the similarity type of Chang's *MV algebras* (see e.g. (7)), which turns out to share nearly all the most relevant properties of MV algebras, with the notable exception that there is no neutral element for $\oplus^{\mathbf{D}}$: in fact, $\langle a, b\rangle \oplus^{\mathbf{D}} \langle c, d\rangle = \langle \min(1, a + c), \frac{1}{2}\rangle \neq \langle a, b\rangle$ whenever $b \neq \frac{1}{2}$. In other words, $\mathbf{D}$ fails to satisfy the equation $x \oplus 0 \approx x$; its equational theory differs from the equational theory of MV algebras. Thus, it makes sense to try and axiomatize it. With this aim in mind, we start by introducing the notion of *quasi-MV algebra* (see (22)).

Definition 4.1:   A *quasi-MV algebra* (for short, qMV algebra) is an algebra $\mathbf{A} = \langle A, \oplus, ', 0, 1\rangle$ of type $\langle 2, 1, 0, 0\rangle$ satisfying the following equations:

A1. $x \oplus (y \oplus z) \approx (x \oplus z) \oplus y$ A5. $(x \oplus 0)' \approx x' \oplus 0$
A2. $x'' \approx x$ A6. $(x \oplus y) \oplus 0 \approx x \oplus y$
A3. $x \oplus 1 \approx 1$ A7. $0' \approx 1$
A4. $(x' \oplus y)' \oplus y \approx (y' \oplus x)' \oplus x$

Of course, a qMV algebra is an MV algebra iff it satisfies the additional equation $x \oplus 0 \approx x$. An immediate consequence of Definition 4.1 is the fact that the class of qMV algebras is a *variety* in its signature. Henceforth, such a variety will be denoted by $q\mathbb{MV}$. The subvariety of MV algebras will be denoted by $\mathbb{MV}$.

As already remarked, every MV algebra is an example of qMV algebra. It is also easy to see that the aforementioned algebra $\mathbf{D}$ is a qMV algebra, henceforth called *standard qMV algebra*.

Two differences between MV algebras and pure qMV algebras are worth noting:

- it is well-known (see e.g. (7)) that it is possible to introduce a *lattice* order on any MV algebra $\mathbf{A}$ by simply taking $a \leq b$ to hold whenever $1 = a' \oplus b$. In our setting, this relation (denoted by $\leq_A$) turns out to be a preordering, but not necessarily a partial ordering (let alone a lattice ordering) of A;
- some elements in a qMV algebra (at least one indeed, i.e. 0) are "well-behaved" in that they satisfy the equation $x \oplus 0 \approx x$; we call them *regular*. Of course, MV algebras contain nothing but regular elements. Pure qMV algebras, on the contrary, also have *irregular* elements which fail to satisfy that equation. We denote by $\mathcal{R}(\mathbf{A})$ the set of all regular elements of $\mathbf{A}$.

The relations $\chi^{\mathbf{A}}$ and $\tau^{\mathbf{A}}$ on A defined by

$$a\chi^{\mathbf{A}}b \text{ iff } a \leq^{\mathbf{A}} b \text{ and } b \leq^{\mathbf{A}} a;$$
$$a\tau^{\mathbf{A}}b \text{ iff } a, b \in \mathcal{R}(\mathbf{A}) \text{ or } a = b$$

are congruences on any qMV algebra $\mathbf{A}$; we drop the superscripts whenever it is clear which algebra is at issue. Moreover, we call *clouds* the elements of A/χ. We have that:

Lemma 4.2: *Let $\mathbf{A}$ be a qMV algebra. The algebra*

$$\mathbf{R_A} = \left\langle \mathcal{R}(\mathbf{A}), \oplus^{\mathbf{R}}, '^{\mathbf{R}}, 0^{\mathbf{R}}, 1^{\mathbf{R}} \right\rangle,$$

where, for any functor f, $f^{\mathbf{R}}$ is the restriction to $\mathcal{R}(\mathbf{A})$ of $f^{\mathbf{A}}$, is an MV-subalgebra of $\mathbf{A}$, lattice ordered by the restriction to $\mathcal{R}(\mathbf{A})$ of $\leq^{\mathbf{A}}$, and isomorphic to $\mathbf{A}/\chi$.

qMV algebras consisting of just one cloud are called *flat*; they correspond to the subvariety $\mathbb{F}q\mathbb{MV}$ of qMV algebras whose equational basis is the single equation $0 \approx 1$. Remark that, for any qMV algebra $\mathbf{A}$, $\mathbf{A}/\chi$ is an MV algebra, while $\mathbf{A}/\tau$ is a flat algebra. But there is more to it: any qMV algebra can be thought of as composed by an MV algebraic component ($\mathbf{A}/\chi$) and a flat component ($\mathbf{A}/\tau$):

Theorem 4.3: *For every qMV algebra $\mathbf{Q}$, there exist an MV algebra $\mathbf{M}$ and a flat qMV algebra $\mathbf{F}$ such that $\mathbf{Q}$ can be embedded into the direct product $\mathbf{M} \times \mathbf{F}$.*

As a corollary to Theorem 4.3, to Chang's completeness theorem for MV algebras and to the completeness of flat qMV algebras with respect to a standard flat algebra over the complex numbers, we get the following completeness result with respect to the standard algebra $\mathbf{D}$:

Theorem 4.4: *If t, s are terms in the language of qMV algebras, $q\mathbb{MV} \vDash t \approx s$ iff $\mathbf{D} \vDash t \approx s$.*

5. Adding square roots of the inverse

We now want to expand quasi-MV algebras by an additional unary operation of *square root of the inverse* (19), corresponding to the $\sqrt{\text{NOT}}$ gate.

Definition 5.1: A $\sqrt{'}$ *quasi-MV algebra* (for short, $\sqrt{'}qMV$ *algebra)* is an algebra $\mathbf{A} = \left\langle A, \oplus, \sqrt{'}, 0, 1, k \right\rangle$ of type $\langle 2, 1, 0, 0, 0 \rangle$ such that, upon defining $a' = \sqrt{'}\sqrt{'}a$ for all $a \in A$, the following conditions are satisfied:

> SQ1. the term reduct $\langle A, \oplus, ', 0, 1 \rangle$ is a quasi-MV algebra;
> SQ2. $k = \sqrt{'}k$
> SQ3. $\sqrt{'}(a \oplus b) \oplus 0 = k$ for all $a, b \in A$.

$\sqrt{'}qMV$ algebras form a variety in their own similarity type, hereafter named $\sqrt{'}q\mathbb{MV}$. We remark in passing that it is impossible to add a square root of the inverse to a nontrivial MV algebra: letting b be 0 in SQ3, for all $a \in A$ we would have $\sqrt{'}a = k$, whence by SQ2 $a' = \sqrt{'}\sqrt{'}a = \sqrt{'}k = k$ and so $a = a'' = k' = \sqrt{'}\sqrt{'}k = k$.

An example of $\sqrt{'}qMV$ algebra is the following term expansion of $\mathbf{D}$; finite examples of $\sqrt{'}qMV$ algebras can be found in (19).

Example 5.2 $\mathbf{D}_r$ is the algebra $\left\langle D, \oplus^{\mathbf{D}_r}, \sqrt{'}^{\mathbf{D}_r}, 0^{\mathbf{D}_r}, 1^{\mathbf{D}_r}, k^{\mathbf{D}_r} \right\rangle$, where:

- $\langle D, \oplus^{\mathbf{D}_r}, '^{\mathbf{D}_r}, 0^{\mathbf{D}_r}, 1^{\mathbf{D}_r} \rangle$ is the above-mentioned qMV algebra $\mathbf{D}$;
- $\sqrt{\prime}^{\mathbf{D}_r} \langle a, b \rangle = \langle b, 1 - a \rangle$;
- $k^{\mathbf{D}_r} = \langle \frac{1}{2}, \frac{1}{2} \rangle$.

In $\sqrt{\prime}$qMV algebras we have not only regular elements, but also *coregular* elements, i.e. elements whose square roots of the inverse are regular. In other words, a is coregular just in case $\sqrt{\prime}a \oplus 0 = \sqrt{\prime}a$. We denote by $\mathcal{COR}(\mathbf{A})$ the set of all coregular elements of $\mathbf{A}$.

In the present setting, the analogues of the qMV-algebraic "crucial" congruences χ and τ are as follows:

Definition 5.3: Let $\mathbf{A}$ be a $\sqrt{\prime}$qMV algebra and let $a, b \in A$. We set:

$$a \lambda^{\mathbf{A}} b \ \text{ iff } \ a \leq^{\mathbf{A}} b, \ b \leq^{\mathbf{A}} a, \ \sqrt{\prime}a \leq^{\mathbf{A}} \sqrt{\prime}b \text{ and } \sqrt{\prime}b \leq^{\mathbf{A}} \sqrt{\prime}a.$$

As one can easily realism, $a \lambda^{\mathbf{A}} b$ iff $a \oplus 0 = b \oplus 0$ and $\sqrt{\prime}a \oplus 0 = \sqrt{\prime}b \oplus 0$. It turns out that $\lambda^{\mathbf{A}}$ is a congruence on every $\sqrt{\prime}$qMV algebra. We call the relation $\lambda^{\mathbf{A}}$ the *cartesian* congruence on a given $\sqrt{\prime}$qMV algebra, and drop once again the superscripts whenever it is clear which algebra is at issue. Likewise, we introduce a congruence which we call the *flat* congruence on a $\sqrt{\prime}$ qMV algebra. Omitting superscripts from the very beginning, we put:

Definition 5.4: Let $\mathbf{A}$ be a $\sqrt{\prime}$qMV algebra and let $a, b \in A$. We define:

$$a \mu b \text{ iff } a = b \text{ or } a, b \in \mathcal{R}(\mathbf{A}) \cup \mathcal{COR}(\mathbf{A})$$

We now introduce two special classes of $\sqrt{\prime}$ qMV algebras: *cartesian* algebras, where λ is the identity, and *flat* algebras, where λ is the universal relation.

Definition 5.5: A $\sqrt{\prime}$ qMV algebra $\mathbf{A}$ is called *cartesian* iff $\lambda = \Delta$, i.e. iff it satisfies the following condition (quasiequation):

$$x \oplus 0 \approx y \oplus 0 \text{ and } \sqrt{\prime}x \oplus 0 \approx \sqrt{\prime}y \oplus 0 \text{ implies } x \approx y.$$

A $\sqrt{\prime}$ qMV algebra $\mathbf{A}$ is called *flat* iff $\lambda = \nabla$. We denote by $\mathbb{F}$ the class of flat $\sqrt{\prime}$ qMV algebras, and by $\mathbb{C}$ the class of cartesian $\sqrt{\prime}$ qMV algebras.

As a consequence of the definition, the only $\sqrt{\prime}$qMV algebra which is both cartesian and flat is the trivial one-element algebra. It is worth noticing that $\mathbb{F}$ is a variety, whose equational basis in $\sqrt{\prime}q\mathbb{MV}$ is given by the single equation $0 \approx 1$, while $\mathbb{C}$ is a quasivariety which is not a variety.

Cartesian $\sqrt{\prime}$qMV algebras are amenable to a clean representation in terms of algebras of *pairs*. We first introduce a suitable construction on MV algebras having a fixpoint for the inverse:

Definition 5.6: Let $\mathbf{A} = \langle A, \oplus^{\mathbf{A}}, '^{\mathbf{A}}, 0^{\mathbf{A}}, 1^{\mathbf{A}} \rangle$ be an MV algebra and let $k \in A$ be such that $k = k'$. The *pair algebra* over $\mathbf{A}$ is the algebra

$$\mathcal{P}(\mathbf{A}) = \left\langle A^2, \oplus^{\mathcal{P}(\mathbf{A})}, \sqrt{\prime}^{\mathcal{P}(\mathbf{A})}, 0^{\mathcal{P}(\mathbf{A})}, 1^{\mathcal{P}(\mathbf{A})}, k^{\mathcal{P}(\mathbf{A})} \right\rangle$$

where:

- $\langle a, b \rangle \oplus^{\mathcal{P}(\mathbf{A})} \langle c, d \rangle = \langle a \oplus^{\mathbf{A}} c, k \rangle$;
- $\sqrt{\prime}^{\mathcal{P}(\mathbf{A})} \langle a, b \rangle = \langle b, a'^{\mathbf{A}} \rangle$;

- $0^{\mathcal{P}(\mathbf{A})} = \langle 0^{\mathbf{A}}, k \rangle$;
- $1^{\mathcal{P}(\mathbf{A})} = \langle 1^{\mathbf{A}}, k \rangle$;
- $k^{\mathcal{P}(\mathbf{A})} = \langle k, k \rangle$.

Every pair algebra $\mathcal{P}(\mathbf{A})$ over an MV algebra $\mathbf{A}$ is a cartesian $\sqrt{'}$qMV algebra. Conversely, every cartesian $\sqrt{'}$qMV algebra is embeddable into a pair algebra via the mapping $f(a) = \left\langle a \oplus 0, \sqrt{'}a \oplus 0 \right\rangle$:

Theorem 5.7: *Every cartesian $\sqrt{'}qMV$ algebra $\mathbf{A}$ is embeddable into the pair algebra $\mathcal{P}(\mathbf{R_A})$ over its MV polynomial subreduct $\mathbf{R_A}$ of regular elements.*

A variant of the direct decomposition for quasi-MV algebras provided by Theorem 4.3 carries over to our enriched structures: any $\sqrt{'}$qMV algebra can be thought of as composed by a cartesian component $(\mathcal{P}(\mathbf{R_Q})$, the pair $\sqrt{'}$qMV algebra over the MV algebra $\mathbf{R_Q}$ of regular elements of $\mathbf{Q})$ and a flat component $(\mathbf{A}/\mu)$:

Theorem 5.8: *For every $\sqrt{'}qMV$ algebra $\mathbf{Q}$, there exist a cartesian algebra $\mathbf{C}$ and a flat algebra $\mathbf{F}$ such that $\mathbf{Q}$ can be embedded into the direct product $\mathbf{C} \times \mathbf{F}$.*

It is shown in (19) that the quasivariety of cartesian $\sqrt{'}$qMV algebras generates the whole variety $\sqrt{'}q\mathbb{MV}$:

Theorem 5.9: $\mathbf{V}(\mathbb{C}) = \sqrt{'}q\mathbb{MV}$.

Moreover, the standard completeness theorem for qMV algebras carries over to $\sqrt{'}q\mathbb{MV}$:

Theorem 5.10: *Let $t, s \in Term(\langle 2, 1, 0, 0, 0 \rangle)$. Then*

$$\mathbf{D}_r \vDash t \approx s \quad \textit{iff } \sqrt{'}q\mathbb{MV} \vDash t \approx s.$$

Among the additional results proved for quasi-MV algebras and $\sqrt{'}$ quasi-MV algebras in (22), (19), (26), (3), we mention the following (whenever we fail to specify for which variety a given result has been proved, it has to be understood that it holds for both $q\mathbb{MV}$ and $\sqrt{'}q\mathbb{MV}$):

- a representation of quasi-MV algebras as labelled MV algebras;
- finite model property; strong finite model property for quasi-MV algebras;
- congruence extension property;
- amalgamation property;
- failure of several algebraic properties, including congruence modularity, subtractivity and point regularity;
- a characterization of free algebras;
- a characterization of quasi-MV term reducts and subreducts of $\sqrt{'}$ quasi-MV algebras;
- a description of the lattice of subvarieties of quasi-MV algebras.

6.   Some drawbacks

The approach outlined so far seems to reconcile within a single framework the different aspects of vagueness and uncertainty referred to in our introductory section: quantum computation, which can be seen as an investigation of the notion of uncertainty in information theory, admits fuzzy-like structures as underlying algebras. A possible criticism to this perspective, however, comes from a limitative result recently proved ((13)).

As usual, by *boolean function* we mean a function $f : \{0,1\}^n \to \{0,1\}$; of course, if $n = 2$, the function is said to be *binary*. A binary *fuzzy function*, on the other hand, is a function $g : [0,1]^2 \to [0,1]$. The key concept to be used in what follows is the notion of *fuzzy extension* of a binary boolean function.

Definition 6.1: Let f be a binary boolean function. The binary fuzzy function g is a *fuzzy extension* of f iff $g \lceil \{0,1\}^2 = f$.

The notion of fuzzy extension of a boolean function is natural enough: by means of it we can partition binary fuzzy functions into "families" modulo their identity of behavior on the endpoints of the closed real unit interval. For example, the family of the fuzzy "conjunctions" can be identified with the fuzzy extensions of boolean conjunction; this class contains as members, e.g., product, Łukasiewicz conjunction, and the min function. Likewise, the family of the fuzzy "disjunctions" will contain Łukasiewicz disjunction, the max function and the MYCIN sum $g(x,y) = x + y - xy$.

An interesting question now arises: which fuzzy extensions of binary boolean functions admit of a quantum computational counterpart? To address this problem properly, we first need to exactly specify what it means for a fuzzy function to have a quantum analogue. The next definition provides what is needed.

Definition 6.2: A binary fuzzy function g is said to be *quantum simulable* iff there exists an $n \geq 1$, a unitary operator U_g on $\overset{n+2}{\bigotimes} \mathbb{C}^2$ and a quregister $|\chi\rangle$ in $\overset{n}{\bigotimes} \mathbb{C}^2$ s.t., for any pair $|\varphi\rangle, |\psi\rangle$ of qubits in $\mathbb{C}^2$, the following condition is satisfied:

$$\mathrm{p}\left(U_g\left(|\varphi\rangle|\psi\rangle|\chi\rangle\right)\right) = g(\mathrm{p}\left(|\varphi\rangle\right), \mathrm{p}\left(|\psi\rangle\right)).$$

In plain words, and with a good deal of oversimplification, we might say that a binary fuzzy function g is quantum simulable whenever there is an associated unitary operator U such that, for any qubits $|\varphi\rangle, |\psi\rangle$, the probability of $U_g\left(|\varphi\rangle|\psi\rangle|\chi\rangle\right)$ is just the result of the application of g to the probabilities of $|\varphi\rangle$ and $|\psi\rangle$. We now have the following result and corollary:

Theorem 6.3: *(13) Let f be a binary boolean function. The fuzzy function g_f defined by:*

$$g_f(x,y) = (1-x)(1-y)f(0,0) + (1-x)yf(0,1) + x(1-y)f(1,0) + xyf(1,1)$$

is the unique *quantum simulable fuzzy extension of f.*

Corollary 6.4: *The MYCIN sum is the unique quantum simulable fuzzy extension of boolean inclusive disjunction.*

It follows from the previous corollary that the Łukasiewicz disjunction is not quantum simulable, a fact which seems to undermine at its very root our approach to the logic of quantum computation, where, as we have seen, a quantum analogue of Łukasiewicz disjunction plays a central role. Is there any way out of this seemingly blind alley? In our opinion, three equally legitimate - and far from being mutually exclusive - attitudes could be taken up when faced with the aforementioned challenge.

- *Biting the bullet.* A plausible response to any motivational objection to the investigation of quasi-MV algebras and their expansions, raised in the light of Corollary 6.4, could still be along the lines of *so what*? After all, these structures are well-motivated enough in that, as we have seen, the truncated sum $\oplus$ corresponds to an operation on projectors which is natural and well-behaved

in terms of probability assignments; furthermore, when combined with ′ it allows to define a relation of *(pre-)order* among density operators which, in its absence, ought to be introduced as a primitive concept. In addition, quasi-MV algebras are ever more gaining an intrinsic universal algebraic interest if viewed as generalizations of MV algebras to the semisubtractive but not point regular case (cp. (3)). Therefore, they remain worth studying even though our ⊕ does not correspond (in the sense specified above) to any unitary operator.

- *Vindicating the Łukasiewicz disjunction.* Recently, the definition of *quantum gate* as a unitary (reversible) operator whose arguments are quregisters has been questioned as exceedingly restrictive. In the presence of certain phenomena, such as decoherence and noise, or intermediate measurements in a computational process which yield mixed states as outputs, it is expedient to resort to a more general notion of *quantum operation*: such operators need not be unitary, while their arguments can be qumixes as well as quregisters (1). The concept of quantum operation hints at a possible way out of the dead end of Theorem 6.3. In fact, we recently proved (17) that, although the Łukasiewicz disjunction on the real unit interval is not quantum simulable, its quantum analogue of Definition 3.8 - which, we emphasis, is not even a quantum operation in the sense of (1) - can nonetheless be approximated by means of (appropriately defined) polynomial quantum operations[1].

- *Exploring new paths.* Were we to aim at an algebraic analysis of quantum analogues of fuzzy operations in the sense of Definition 6.2, we should plausibly focus on an investigation of the algebraic properties of the unit square $[0,1]^2$ endowed with quantum analogues of product and of the MYCIN sum, as well as with the unary operations of inverse and square root of the inverse. Here the main problem seems to be that the ground looks bumpy even on the fuzzy logical side. In fact, while MV algebras with all of their rich structure theory - and especially with Chang's completeness theorem - have directed our efforts with quasi-MV algebras, in the present case there is nothing comparable to rely on. True, there exists a lot of work both on Hajek's product logic (see e.g. (21)) and on appropriate combinations of Łukasiewicz and product logic (we just mention (24)), and there is even a result to the effect that Łukasiewicz disjunction is definable within product logic enriched with Łukasiewicz negation ((8)). However, the latter result heavily depends on the presence of an implication residuating product, and such an implication fails to be a quantum simulable fuzzy extension of material implication. An acceptable analysis of this kind of structures is beyond the state of the art, but is definitely one of the major goals of our activity as a research group.

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[1]This "Stone-Weierstrass-type" result is actually much more general: every continuous function $f : (0,1)^n \to (0,1)$ has a quantum analogue which can be approximated in such a way by means of polynomial quantum operations.

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