Fredkin and Toffoli quantum gates: fuzzy representations and comparison

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In the framework of quantum computation with mixed states, fuzzy representations based on continuous t-norms for Toffoli and Fredkin quantum gates are introduced. A comparison between both gates is also studied.

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Introduction

Standard quantum computing is based on quantum systems described by finite dimensional Hilbert spaces, specially $\mathbb{C}^2$, that is the two-dimensional space where qbits live. A qubit (the quantum counterpart of the classical bit) is represented by a unit vector in $\mathbb{C}^2$ and, generalizing for a positive integer n, n-qubits are represented by unit vectors in $\otimes^n \mathbb{C}^2$. Similarly to the classical case, it is possible to study the behavior of a number of quantum logical gates (hereafter quantum gates, for short) operating on qbits. These quantum gates are represented by unitary operators.

In[3,5] a quantum gate system based on Toffoli gate is studied. This system is interesting for two main reasons: (i) it is related to continuous t-norms,[14] i.e., continuous binary operations on the interval $[0,1]$ that are commutative, associative, non-decreasing and with 1 as the unit element. They are naturally proposed in fuzzy logic as interpretations of the conjunction.[13] (ii) A generalization of the mentioned system to mixed states allows us to connect it with sequential effect algebras,[10] introduced to study the sequential action of quantum effects which are unsharp versions of quantum events.[11,12] However there exists another quantum gate, the Fredkin gate, whose behavior is similar to the Toffoli gate. Moreover, under particular conditions, it allows us to represent the same continuous t-norms that Toffoli gate. It suggests to introduce a comparison between Toffoli and Fredkin gates.

The aim of this paper is to study a probabilistic type representation of Toffoli and Fredkin gates based on Łukasiewicz negation $\neg x = 1 - x$, Łukasiewicz sum $x \oplus y = \min\{x + y, 1\}$ and Product t-norms $x \cdot y$ in the framework of quantum computation with mixed states and to establish a comparison between both. Note that, the interval $[0,1]$ equipped with the operations $\langle \oplus, \cdot, \neg \rangle$, defines an algebraic structure called *product* MV*-algebra* (PMV-algebra for short).[17] In our representation, circuits made from assemblies of Toffoli and Fredkin gates, can be probabilistically represented as $\langle \oplus, \cdot, \neg \rangle$-polynomial expressions in a PMV-algebra. In this way, PMV-algebra structure related to Toffoli and Fredkin gates, plays a similar role than Boolean algebras describing digital circuits.

The paper is organized as follows: in Section 1 we introduce basic notions of quantum computational logic and we fix some mathematical notation. In Section 2 we briefly describe the Controlled Unitary Operations, that turn out to be very useful in the rest of the paper. In Section 3 and in Section 4 fuzzy representations related to Toffoli and Fredkin gates are respectively provided. In Section 5 we make a comparison between Toffoli and Fredkin gate.

1. Basic notions

In quantum computation, information is elaborated and processed by means of quantum systems. Pure states of a quantum system are described by unit vectors in a Hilbert space. A *quantum bit* or *qbit*, the fundamental concept of quantum computation, is a pure state in the Hilbert space $\mathbb{C}^2$. The *standard orthonormal basis* $\{|0\rangle, |1\rangle\}$ of $\mathbb{C}^2$ is generally called *quantum computational basis*. Intuitively, $|1\rangle$ is related to the truth logical value and $|0\rangle$ to the falsity. Thus, pure states $|\psi\rangle$ in $\mathbb{C}^2$ are superpositions of the basis vectors with complex coefficients: $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$, where $|c_0|^2 + |c_1|^2 = 1$.

In the usual representation of quantum computational processes, a quantum circuit is identified with an appropriate composition of *quantum gates*, mathematically represented by *unitary operators* acting on pure states of a convenient (n-fold tensor product) Hilbert space $\otimes^n\mathbb{C}^2$.[18] A special basis, called the 2^n-*standard orthonormal basis*, is chosen for $\otimes^n\mathbb{C}^2$. More precisely, it consists of the 2^n-orthogonal states $|\iota\rangle$, $0 \leq \iota \leq 2^n$ where ι is in binary representation and $|\iota\rangle$ can be seen as the tensor product of states $|\iota\rangle = |\iota_1\rangle \otimes |\iota_2\rangle \otimes \ldots \otimes |\iota_n\rangle$, where $\iota_j \in \{0, 1\}$. It provides the standard quantum computational model, based on qbits and unitary operators.

In general, a quantum system is not in a pure state. This may be caused, for example, by the non-complete efficiency in the preparation procedure or by the fact that systems can not be completely isolated from the environment, undergoing decoherence of their states. On the other hand, there are interesting processes that can not be encoded in unitary evolutions. For example, at the end of the computation a non-unitary operation - a measurement - is applied, and the state becomes a probability distribution over pure states, or what is called a *mixed state*. In view of these facts, several authors[1,5,7,8,10] have paid attention to a more general model of quantum computational processes, where pure states are replaced by mixed states. In what follows we give a short description of this mathematical model.

To each vector of the quantum computational basis of $\mathbb{C}^2$ we may associate two density operators $P_0 = |0\rangle\langle 0|$ and $P_1 = |1\rangle\langle 1|$ that represent the standard basis in this framework. Let $P_1^{(n)}$ be the operator $P_1^{(n)} = (\otimes^{n-1}I) \otimes P_1$ on $\otimes^n\mathbb{C}^2$, where I is the 2×2 identity matrix. Clearly, $P_1^{(n)}$ is a 2^n-square matrix. By applying the Born rule, we consider the probability of a density operator ρ as follows:

$$p(\rho) = tr(P_1^{(n)}\rho) \tag{1}$$

We focus our attention in this probability values since it allows us to establish a link between Toffoli gate and fuzzy connectives. Note that, in the particular case in which $\rho = |\psi\rangle\langle\psi|$ where $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$, we obtain $p(\rho) = |c_1|^2$. Thus, this probability value associated to ρ is the generalization, in this model, of the probability that a measurement over $|\psi\rangle$ yields $|1\rangle$ as output. A *quantum operation*[15] is a linear operator $\mathcal{E} : \mathcal{L}(H_1) \to \mathcal{L}(H_2)$ where $\mathcal{L}(H_i)$ is the space of linear operators in the complex Hilbert space H_i $(i = 1, 2)$, representable as $\mathcal{E}(\rho) = \sum_i A_i \rho A_i^\dagger$, where A_i are operators satisfying $\sum_i A_i^\dagger A_i = I$ (Kraus representation[15]). It can be seen that a quantum operation maps density operators into density operators. Each unitary operator U gives rise to a quantum operation $\mathcal{O}_U$ such that $\mathcal{O}_U(\rho) = U\rho U^\dagger$ for any density operator ρ. In the case in which U is a real unitary operator, then probability of $\mathcal{O}_U(\rho)$ is simply given by

$$p(\mathcal{O}_U) = tr(P_1^{(n)} \cdot U\rho U) = tr((UP_1^{(n)}U) \cdot \rho). \quad (2)$$

The model based on density operators and quantum operations is called "*quantum computation with mixed states*". It allows us to also represent irreversible processes as measurements in the middle of the computation.

The connection between a quantum operation $\mathcal{E}$ and continuous t-norms arises when the generic probability values $p(\mathcal{E}(-\otimes \ldots \otimes -)$ can be described in terms of the operations $\langle \oplus, \cdot, \neg \rangle$ defined in the introduction.

Let us define a $\langle \oplus, \cdot, \neg \rangle_n$-*polynomial expression* as a function $f : [0, 1]^n \to [0, 1]$ built only using the three operations $\langle \oplus, \cdot, \neg \rangle$ and n variables.

Now we can formally introduce the connection between quantum operations and continuous t-norms.

Definition 1.1. Let $\mathcal{E} : \mathcal{L}(\otimes^m \mathbb{C}^2) \to \mathcal{L}(\otimes^r \mathbb{C}^2)$ be a quantum operation. Then $\mathcal{E}$ is said to be $\langle \oplus, \cdot, \neg \rangle_n$-representable if and only if there exists a $\langle \oplus, \cdot, \neg \rangle_n$-*polynomial expression* $f : [0, 1]^n \to [0, 1]$ and natural numbers $k_1, \ldots k_n$ satisfying $k_1 + \ldots + k_n = m$, such that:

$$p(\mathcal{E}(\rho_1 \otimes \ldots \otimes \rho_n)) = f(p(\rho_1), \ldots, p(\rho_n))$$

where ρ_i is a density operator in $\otimes^{k_i} \mathbb{C}^2$.

This definition turns out to be crucial in the fuzzy representations of Toffoli and Fredkin gates provided in Section 3 and Section 4, respectively.

2. Controlled Unitary Operators

By following the standard construction of controlled operators (see, *e.g.*, section 4.3 in[18]), if $U^{(l)}$ is a unitary l-qubit gate, then the controlled-U

gate operating on $l+1$ qubits assumes the following block-representation:

$$CU^{(1,l)} = \left[\begin{array}{c|c} I^{(l)} & 0 \\ \hline 0 & U^{(l)} \end{array}\right].$$

This block representation allows us to end up with the following operational form of an arbitrary $CU^{(1,l)}$ gate:

$$\begin{aligned} CU^{(1,l)} &= \left[\begin{array}{c|c} I^{(l)} & 0 \\ \hline 0 & U^{(l)} \end{array}\right] = \left[\begin{array}{c|c} I^{(l)} & 0 \\ \hline 0 & 0 \end{array}\right] + \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & U^{(l)} \end{array}\right] \\ &= P_0 \otimes I^{(l)} + P_1 \otimes U^{(l)}. \end{aligned}$$

Further, the generalized control unitary $CU^{(m,l)}$ gate is given by:

$$CU^{(m,l)} = I^{(m-1)} \otimes \left[\begin{array}{c|c} I^{(l)} & 0 \\ \hline 0 & U^{(l)} \end{array}\right] \;\; = P_0^{(m)} \otimes I^{(l)} + \; P_1^{(m)} \otimes U^{(l)} \qquad (3)$$

$$= I^{(m-1)} \otimes \left(P_0 \otimes I^{(l)} + P_1 \otimes U^{(l)}\right) = I^{(m-1)} \otimes CU^{(1,l)}. \qquad (4)$$

As a useful example, in the special case where the unitary operator U is the well known Not gate defined as $Not = \begin{bmatrix} 0 \; 1 \\ 1 \; 0 \end{bmatrix}$ and whose extension to higher dimensions is given by: $Not^{(l)} \;=\; I^{(l-1)} \otimes Not$, then the notion of $Control - Not$ gate $CNot^{(m,l)}$ is given by :

$$CNot^{(m,l)} \;=\; P_0^{(m)} \otimes I^{(l)} \;+\; P_1^{(m)} \otimes Not^{(l)}. \qquad (5)$$

3. Fuzzy representation of Toffoli gate

The Toffoli gate, introduced by Tommaso Toffoli,[21] is a universal reversible logical gate, which means that any classical reversible circuit can be built from an ensemble of Toffoli gates. This gate has three input bits (x, y, z) and three output bits. Two of the bits, x and y, are control bits that are unaffected by the action of the gate. The third bit z is the target bit that is flipped if both control bits are set to 1, and otherwise is left unchanged. The application of the Toffoli gate to a set of three bits is dictated by:

$$T(x, y, z) = (x, y, xy \widehat{+} z)$$

where $\widehat{+}$ is the sum modulo 2. The Toffoli gate can be used to reproduce the classical AND gate when $z = 0$ and the $NAND$ gate when $z = 1$.

The classical definition of the Toffoli gate can extended as a quantum gate in the following way.

Definition 3.1. For any natural numbers $n, m, l \geq 1$ and for any vectors of the standard orthonormal basis $|x\rangle = |x_1 \dots x_n\rangle \in \otimes^n \mathbb{C}^2$, $|y\rangle = |y_1 \dots y_m\rangle \in \otimes^m \mathbb{C}^2$ and $|z_1 \dots z_l\rangle \in \otimes^l \mathbb{C}^2$, the Toffoli quantum gate $T^{(m,n,l)}$ (from now on, shortly, Toffoli gate) on $\otimes^{n+m+l}\mathbb{C}^2$ is defined as follows:

$$T^{(n,m,l)}(|x\rangle \otimes |y\rangle \otimes |z\rangle) = |x\rangle \otimes |y\rangle \otimes |x_n y_m \widehat{+} z_l\rangle.$$

Taking into account that the Toffoli gate can be interpreted as a *Control-Control-Not gate*,[9] we have that:

$$\begin{aligned}
T^{(n,m,l)} &= CCNot^{(n,m,l)} = I^{(n-1)} \otimes \left[\begin{array}{c|c} I^{(m+l)} & \mathbf{0} \\ \hline \mathbf{0} & CNot^{(m,l)} \end{array}\right] \\
&= P_0^{(n)} \otimes I^{(m+l)} + P_1^{(n)} \otimes CNot^{(m,l)} \\
&= (I^{(n)} - P_1^{(n)}) \otimes I^{(m+l)} + P_1^{(n)} \otimes \Big((I^{(m)} - P_1^{(m)}) \otimes I^{(l)} + P_1^{(m)} \otimes Not^{(l)} \Big) \\
&= (I^{(n+m)} - P_1^{(n)} \otimes P_1^{(m)}) \otimes I^{(l)} + P_1^{(n)} \otimes P_1^{(m)} \otimes Not^{(l)} = \\
&= I^{(n+m+l)} + P_1^{(m)} \otimes P_1^{(m)} \otimes (Not^{(l)} - I(l)) \; .
\end{aligned}$$

The following Theorem provides a fuzzy representation founded of the probability value of the Toffoli gate.

Theorem 3.1. *Let ρ, σ, τ be density operators such that $\rho \in \otimes^n \mathbb{C}^2$, $\sigma \in \otimes^m \mathbb{C}^2$ and $\tau \in \otimes^l \mathbb{C}^2$. Then*

$$p(T^{(n,m,l)}(\rho \otimes \sigma \otimes \tau)T^{(n,m,l)}) = (1 - p(\tau))p(\rho)p(\sigma) + p(\tau)(1 - p(\rho)p(\sigma))$$

and the quantum operation associated to $T^{(m,n,l)}$ is $\langle \oplus, \cdot, \neg \rangle_3$-representable by $\neg z \cdot x \cdot y \quad \oplus \quad z \cdot \neg(x \cdot y)$.

Proof.

$$\begin{aligned}
&p(T^{(n,m,l)}(\rho \otimes \sigma \otimes \tau)T^{(n,m,l)}) = \\
&= tr(P_1^{(n+m+l)}T^{(n,m,l}(\rho \otimes \sigma \otimes \tau)T^{(n,m,l}) = \\
&= tr(P_1^{(n,m,l)}((I^{n+m} - I^{(n)} \otimes P_1^{(m)}) \otimes I^{(l)} + \\
&+ P_1^{(n)} \otimes P_1^{(m)} \otimes Not^{(l)})(\rho \otimes \sigma \otimes \tau) \cdot ((I^{(n+m)} - P_1^{(n)} \otimes P_1^{(m)}) \otimes I^{(l)} + P_1^{(n)} \otimes P_1^{(m)} \otimes Not^{(l)})) = \\
&= tr(((I^{(n+m)} - I^{(n)} \otimes P_1^{(m)}) \otimes P_1^{(l)} + \\
&+ P_1^{(n)} \otimes P_1^{(m)} \otimes P_1^{(l)} Not^{(l)})(\rho \otimes \sigma \otimes \tau) \cdot \\
&\cdot \ ((I^{(n+m)} - P_1^{(n)} \otimes P_1^{(m)}) \otimes I^{(l)} + P_1^{(n)} \otimes P_1^{(m)}) \otimes Not^{(l)})) = \\
&= tr(((I^{(n+m)} - P_1^{(n)} \otimes P_1^{(m)}) \otimes I^{(l)} + P_1^{(n)} \otimes P_1^{(m)} \otimes Not^{(l)}) \\
&\quad ((I^{(n+m)} - I^{(n)} \otimes P_1^{(m)}) \otimes P_1^{(l)} + \\
&+ P_1^{(n)} \otimes P_1^{(m)} \otimes P_1^{(l)} Not^{(l)}(\rho \otimes \sigma \otimes \tau)) = \\
&= tr(((I^{(n+m)} - P_1^{(n)} \otimes P_1^{(m)}) \otimes P_1^{(l)} + \\
&+ P_1^{(n)} \otimes P_1^{(m)} \otimes Not^{(l)} P_1^{(l)} Not^{(l)}(\rho \otimes \sigma \otimes \tau)) = \\
&= tr((I^{(n+m)} - P_1^{(n)} \otimes P_1^{(m)})(\rho \otimes \sigma) \otimes P_1^{(l)}\tau) + \\
&+ tr(P_1^{(m)}\rho \otimes P_1^{(m)}\rho \otimes P_0^{(l)}\tau) = \\
&= tr((I^{(n+m)} - P_1^{(n)} \otimes P_1^{(m)})(\rho \otimes \sigma))tr(P_1^{(l)}\tau) + \\
&+ tr(P_1^{(n)}\rho)tr(P_1^{(m)}\sigma)tr(P_0^{(l)}\tau) = \\
&= (1 - p(\rho)p(\sigma))p(\tau) + p(\rho)p(\sigma)(1 - p(\tau)).
\end{aligned}$$

Since $p(T^{(n,m,l)}(\rho \otimes \sigma \otimes \tau)T^{(n,m,l)}) \leq 1$*, then the expression* $(1 - p(\rho)p(\sigma))p(\tau) + p(\rho)p(\sigma)(1-p(\tau)) = (1-p(\rho)p(\sigma))p(\tau) \oplus p(\rho)p(\sigma)(1-p(\tau))$*. In this way, we simply obtain that the quantum operation associated to* $T^{(m,n,l)}$ *is* $\langle \oplus, \cdot, \neg \rangle_3$*-representable by* $\neg z \cdot x \cdot y \ \oplus \ z \cdot \neg(x \cdot y)$. □

4. Fredkin gate and its fuzzy representation

The Fredkin gate, introduced by Edward Fredkin,[6] is another example of *universal reversible classical logic gate.*

Also Fredkin is a ternary gate, implementing a *Controlled-Swap* operation. More precisely, let (x, y, z) be a 3-bits input state. The first bit, say x, is taken to be the control bit, remaining unaffected by the action of the gate. The second and the third bits, say y and z, are the target bits that are swapped if the control bit x is set to 1; they remain unchanged otherwise. Fomally:

$$F(x, y, z) = (x,\ y \,\widehat{+}\, x(y \,\widehat{+}\, z),\ z \,\widehat{+}\, x(y \,\widehat{+}\, z)) \ , \tag{6}$$

where, once again, $\widehat{+}$ is the *addition modulo* 2 (equivalent to the XOR operation of the classical sharp logic).

Let us notice that the Fredkin can reproduce the classical AND gate (*i.e.*, when $z_{in} = 0$, $z_{out} = x_{in} \cdot y_{in}$), the classical NOT gate (*i.e.*, when $y_{in} = 0, z_{in} = 1$ $z_{out} = x_{in}\widehat{+}1$) and the classical OR gate (*i.e.*, when $z_{in} = 1$ then y_{out} is the OR between x_{in} and y_{in}).

Also the Fredkin gates can also be naturally extended as a quantum gate in the following way.

Definition 4.1. Let $|x\rangle = |x_1, x_2, \ldots, x_n\rangle$, $|y\rangle = |y_1, y_2, \ldots, y_m\rangle$ and $|z\rangle = |z_1, z_2 \ldots, z_l\rangle$ be vectors of the standard orthonormal basis in $\otimes^n\mathbb{C}^2$, $\otimes^m\mathbb{C}^2$ and $\otimes^l\mathbb{C}^2$, respectively. Then, the quantum Fredkin gate is defined by the following equation:

$$F^{(n,m,l)}|x, y, z\rangle = |x\rangle|y_1 \ldots y_{m-1},\ y_m \widehat{+} x_n(y_m\widehat{+}z_l)\rangle|z_1 \ldots z_{l-1},\ z_l \widehat{+} x_n(y_m\widehat{+}z_l)\rangle.$$

We also notice that, similarly to the Toffoli gate, also the Fredkin gate is a control unitary gate. Hence, it can be represented by using the argument given in Section 2. This unitary gate is the quantum $SWAP^{(m,l)}$ gate.

Note that $SWAP^{(m\,,\,l)}$ is a linear operator that swaps the last qubit (*i.e.*, m^{th} qubit) of the its first input with the last qubit (i.e., l^{th} bit) of its second input.[18,22] Formally, for every state $|y_1, \ldots, y_m, z_1, \ldots, z_l\rangle$ of the computational basis:

$$SWAP^{(m\,,\,l)}|y_1, \ldots, y_m\rangle|z_1, \ldots, z_l\rangle = |y_1, \ldots, y_{m-1}, z_l\rangle|z_1, \ldots, z_{l-1}, y_m\rangle\ . \tag{7}$$

In order to introduce a matrix representation of the $F^{(n,m,l)}$ gate, we first need to provide a matrix form of the $SWAP^{(m,l)}$ gate.

$$SWAP^{(1\,,\,1)} = \begin{bmatrix} 1\,0\,0\,0 \\ 0\,0\,1\,0 \\ 0\,1\,0\,0 \\ 0\,0\,0\,1 \end{bmatrix} = \left[\begin{array}{c|c} P_0 & L_1 \\ \hline L_0 & P_1 \end{array}\right], \tag{8}$$

where L_1 and L_0 are given by $L_1 \equiv |1\rangle\langle 0|$ and $L_0 \equiv |0\rangle\langle 1|$, respectively [a]. These operators can be extended to higher dimensions as $L_1^{(l)} = I^{(l-1)} \otimes L_1$ and $L_0^{(l)} = I^{(l-1)} \otimes\ L_0$, respectively. Hence, we end up with the following

[a] L_1 and L_0 are well known in atomic physics as *Ladder-raising* and the *Ladder-lowering*, respectively.

generalization of the *Swap gate* $SWAP^{(m,l)}$:

$$SWAP^{(m\,,\,l)} \;=\; I^{(m-1)} \otimes SWAP^{(1\,,\,l)} \;=\; I^{(m-1)} \otimes \left[\begin{array}{c|c} P_0^{(l)} & L_1^{(l)} \\ \hline L_0^{(l)} & P_1^{(l)} \end{array}\right]. \qquad (9)$$

By referring to Eq.(3) we easily obtain the generalized quantum Fredkin gate $F^{(n,m,l)}$ as follows.

$$\begin{aligned} F^{(n,m,l)} \;&=\; CSwap^{(n,m,l)} \;=\; I^{(n-1)} \otimes \left[\begin{array}{c|c} I^{(m+l)} & \mathbf{0} \\ \hline \mathbf{0} & SWAP^{(m\,,\,l)} \end{array}\right] && (10) \\ &=\; P_0^{(n)} \otimes I^{(m+l)} \;+\; P_1^{(n)} \otimes SWAP^{(m\,,\,l)} && (11) \\ &=\; P_0^{(n)} \otimes I^{(m+l)} \;+\; P_1^{(n)} \otimes I^{(m-1)} \otimes \left[\begin{array}{c|c} P_0^{(l)} & L_1^{(l)} \\ \hline L_0^{(l)} & P_1^{(l)} \end{array}\right] && (12) \\ &=\; I^{(n+m+l)} \;+\; P_1^{(n)} \otimes \Big(SWAP^{(m\,,\,l)} - I^{(m+l)}\Big). && (13) \end{aligned}$$

The following Theorem provides a fuzzy representation founded of the probability value of the Fredkin gate.

Theorem 4.1. *Let* ρ, σ, τ *be density operators such that* $\rho \in \otimes^n \mathbb{C}^2$, $\sigma \in \otimes^m \mathbb{C}^2$ *and* $\tau \in \otimes^l \mathbb{C}^2$. *Then*

$$p(F^{(n,m,l)}(\rho \otimes \sigma \otimes \tau)F^{(n,m,l)}) = (1 \,-\, p(\rho))\; p(\tau) + p(\rho)\, p(\sigma)$$

and the quantum operation associated to $F^{(m,n,l)}$ *is* $\langle \oplus, \cdot, \neg \rangle_3$*-representable by* $\neg x \cdot z \oplus x \cdot y$.

Proof. By using the matrix representation of $F^{(n,m,l)}$, we obtain:

$$F^{(n,m,l)} \cdot P_1^{(n+m+l)} \cdot F^{(n,m,l)} =$$

$$= I^{(n-1)} \otimes \Big[\Big(P_0 \otimes I^{(m+l)} \ + \ P_1 \otimes SWAP^{(m\,,\,l)}\Big) \cdot \Big(I^{(m+l)} \otimes P_1\Big) \cdot \Big(P_0 \otimes I^{(m+l)} \ + \ P_1 \otimes SWAP^{(m\,,\,l)}\Big)\Big]$$

$$= I^{(n-1)} \otimes \left[\begin{array}{l} ((P_0 \otimes I^{(m+l)}) \cdot (I^{(m+l)} \otimes P_1) \cdot (P_0 \otimes I^{(m+l)})) \\ \ + \ (P_0 \cdot I \cdot P_1) \otimes (\ldots) \\ \ + \ (P_1 \cdot I \cdot P_0) \otimes (\ldots) \\ \ + \ (P_1 \cdot I \cdot P_1) \otimes (I^{(m-1)} \cdot I^{(m-1)} \cdot I^{(m-1)}) \otimes (SWAP^{(1\,,\,l)} \cdot (I \otimes P_1^{(l)}) \cdot SWAP^{(1\,,\,l)}) \end{array}\right]$$

Let us recall that, $P_0 \cdot I \cdot P_1 = P_1 \cdot I \cdot P_0$ that correspond to the null matrix 0.

Further, $SWAP^{(m\,,\,l)} \ = \ I^{(m-1)} \otimes SWAP^{(1\,,\,l)}$.

$$= I^{(n-1)} \otimes \left[\begin{array}{l} (P_0 \cdot I \cdot P_0) \otimes (I^{(m+l-1)} \cdot P_1^{(m+l-1)} \cdot I^{(m+l-1)}) \\ + \ 0 \\ + \ 0 \\ + \ (P_1 \cdot I \cdot P_1) \otimes (I^{(m-1)} \cdot I^{(m-1)} \cdot I^{(m-1)}) \otimes (SWAP^{(1\,,\,l)} \cdot (I \otimes P_1^{(l)}) \cdot SWAP^{(1\,,\,l)}) \end{array}\right]$$

Let us recall that, for density matrices A, B, C, D of appropriate dimension, is

$SWAP^{(m\,,\,l)} \cdot \Big(A^{(m-1)} \otimes B \otimes C^{(l-1)} \otimes D\Big) \cdot SWAP^{(m\,,\,l)} \ = \ A^{(m-1)} \otimes D \otimes C^{(l-1)} \otimes B$. Hence,

$$= I^{(n-1)} \otimes \Big[P_0 \otimes P_1^{(m+l)} + P_1 \otimes I^{(m-1)} \otimes P_1 \otimes I^{(l)}\Big]$$

$$= P_0^{(n)} \otimes I^{(m)} \otimes P_1^{(l)} \ + \ P_1^{(n)} \otimes P_1^{(m)} \otimes I^{(l)}$$

$$= \Big(I^{(n)} - P_1^{(n)}\Big) \otimes I^{(m)} \otimes P_1^{(l)} \ + \ P_1^{(n)} \otimes P_1^{(m)} \otimes I^{(l)}.$$

Therefore, by Eq.(2), the probability value of $F^{(n,m,l)}(\rho \otimes \sigma \otimes \tau)F^{(n,m,l)}$ is given by:

$$p(F^{(n,m,l)}(\rho\otimes\sigma\otimes\tau)F^{(n,m,l)}) = \ Tr\Big[\Big(\Big(I^{(n)} - P_1^{(n)}\Big) \otimes I^{(m)} \otimes P_1^{(l)} \ + \ P_1^{(n)} \otimes P_1^{(m)} \otimes I^{(l)}\Big) \cdot (\rho\otimes\sigma\otimes\tau)\Big]$$

which can be reduced in a straightforward manner to $(1 \, - \, p(\rho)) \; p(\tau) \; + \; p(\rho) \, p(\sigma)$.

Since $p(F^{(n,m,l)}(\rho \otimes \sigma \otimes \tau)F^{(n,m,l)}) \leq 1$, then the expression $(1 \, - \, p(\rho)) \; p(\tau) \; + \; p(\rho) \, p(\sigma) = (1 \, - \, p(\rho)) \; p(\tau) \; \oplus \; p(\rho) \, p(\sigma)$. In this way, we have that the quantum operation associated to $F^{(m,n,l)}$ is $\langle \oplus, \cdot, \neg \rangle_3$-representable by $\neg x \cdot z \oplus x \cdot y$. □

5. Comparing the Toffoli and Fredkin Quantum Gates

In this Section we show are both Toffoli and Fredkin gate are able to represent the product t-norm. However, from a physical point of view, Fredking gate turns out to be more efficient.

An immediate consequence of the Theorem 3.1 and the Theorem 4.1 is that in the special case where $\tau = P_0$ then, for any $\rho \in \otimes^m\mathbb{C}^2, \sigma \in \otimes^n\mathbb{C}^2, \tau \in \mathbb{C}^2$ is $p(T^{(n,m,1)}(\rho \otimes \sigma \otimes P_0)T^{(n,m,1)}) = p(F^{(n,m,1)}(\rho \otimes \sigma \otimes P_0)F^{(n,m,1)}) = p(\rho) \cdot p(\sigma)$. It shows that both quantum gates represent the product t-norm.

A crucial feature of the classical Fredkin gate, in contrast to the Toffoli gate, is that the Fredkin gate is *logically conservative*. This means to say that the number of 1's present in the output of the gate is the same as the number of $1'$s in its input. In other words, the *parity of bits* remains unchanged during the operation of logically-conservative gates like the Fredkin Gate.[4,19–21]

This aspect turns out to be advantageous for building computational circuits that could dissipate less energy (in comparison to the circuits of non-conservative gates) during their operational cycles.[4,19] This is in the light of the well known *Landauer's principle*, according to which, *there is an unavoidable heat-dissipation-cost associated with every bit of information that gets erased*. The theoretical lower bound to the heat-generation of this type is argued to be $K_B T \log 2$. This link between the thermodynamical reversibility and the logical conservativity is due to the well known *statistical-inviolability* of the second law of thermodynamics[2,4,16,19] following a deconstruction of the much debated *Maxwell's demon*.

This crucial aspect of the classical conservative gates gets extended to the quantum gates as follows. Firstly, at a design level, if a gate-module in a given circuit is logically irreversible (meaning that the information encoded by the input states is not entirely recoverable by using the output states alone), then it must be the case that some information about the input states is lost from the gate-module in question. But, the problem of weather this information is irreversibly lost or not, depends on the details of the physical implementation of the gate: this information may either be irreversibly lost –resulting in heat-dissipation, or be just *hidden away* (in a *deterministically retrievable* manner) in some other module of the physical circuit. In a similar case, it may not result in a heat generation, but perhaps costing a memory-resource overhead. The *logically reversible gates* would naturally avoid this type of dissipation at the very design-level itself by leaving a one-to-one correspondence between the output and the input, thereby keeping all the information about the inputs within the same specific gate-module.

However, at the level of physical implementation, there is a further possibility that the operational cycles of even a reversible gate would involve an erasure of some bits of information. There are several factors which could contribute to this information-erasure resulting in dissipation. The main possible reason for this is as follows: though, in theory, all the memory-states are ideally expected to be equally probable, at any non-zero temperature, the memory states of a physical device would be unequally populated following a Boltzman-distribution. This is especially the case of those quantum systems in which the encoding is done onto the energy states of a quantum system. Therein, in ambient temperatures, the ground state is highly populated and the exited states are less populated, following a Boltzman distribution. Further, there is also a natural loss of population from the exited states (also called as the spontaneous emission). This would make it necessary that a standard repumping mechanism be incorporated to retain the memory-states that are encoded using the excited states. These factors would summarily result in an asymmetry in the operational (thermo-economical) cost of different memory-states belonging to the same physical system.

However, there is a possible way of circumventing the above type of dissipation by using those family gates which are not only *logically-reversible* but also *logically-conservative*, like *e.g.,* the Fredkin gate. The strategy is to use an encoding of information such that the most recurring bits of an input is mapped to the most-stable states respectively. Then, the conservativity of the Fredkin gate would guarantee that the number of excited states remains unaltered throughout the operational cycles of the gate: the output would have the same number of excited states as the input was, and hence no extra stabilization cost is required.

Thusly, following the above arguments, even though all the quantum gates by construction are reversible, it becomes desirable to design the circuits based on the logically conservative gates like the Fredkin gate that has been characterized in the present work.

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