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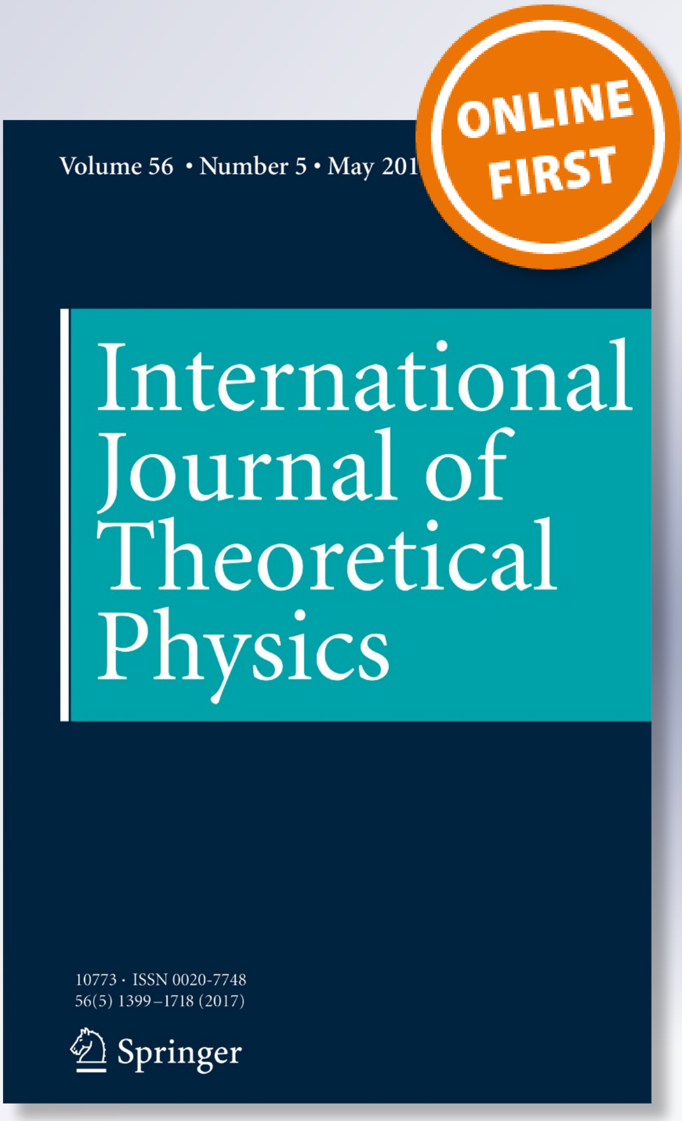

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Fuzzy Type Representation of the Fredkin Gate in Quantum Computation with Mixed States

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Abstract In this work, we introduce a fuzzy type representation of a quantum version of the Fredkin gate in the framework of quantum computation with mixed states.

Keywords Fredkin gate · Density operators · Continuous t-norms

1 Introduction

Standard quantum computing is based on quantum systems described by finite dimensional Hilbert spaces, starting from $\mathbb{C}^2$, that is the two-dimensional space where any qubit (state) lives. A qubit is represented by a unit vector in $\mathbb{C}^2$, while n-qubits are represented by unit vectors in $\mathbb{C}^{2^n}$. Similarly to the classical case, one can introduce a number of quantum logical gates operating on n-qubits, and study their behavior. Similarly to a digital circuit of classical computation, a quantum circuit is modeled as an appropriate composition of quantum logical gates, representable by unitary operators defined over Hilbert spaces of appropriate dimension.

There have been several studies of the fuzzy representations of quantum gates, especially based on continuous t-norms [1, 6–8, 11, 12, 15, 16]. In these studies, it was also discussed that the quantum circuits could be representable using the polynomial-terms of suitable algebraic structures. It is emphasized that these algebraic structures play a similar role that the Boolean algebra in classical computation.

Continuing these investigations, the aim of this paper is to study a fuzzy representation of the Fredkin gate based on the *Łukasiewicz negation* $\neg x = (1 - x)$, the *Łukasiewicz*

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sum $x \oplus y = \min\{x + y, 1\}$ and the *product t-norms* $(x \cdot y)$ in the framework of quantum computation with mixed states. It should be noted that the interval [0, 1] equipped with the operations $\langle \oplus, \cdot, \neg \rangle$, defines an algebraic structure called the *product MV-algebra* (PMV-algebra for short) [9, 20]. Consequently, the circuits made from ensemble of Fredkin gates could be modeled as $\langle \oplus, \cdot, \neg \rangle$-polynomial expressions in a product MV-algebra.

The paper is organized as follows. In Section 2, the basic notions of quantum computation with mixed states are recalled. In Section 3, the family of Fredkin gates is introduced and its implementational advantages are briefly discussed. In Section 4 a useful matrix form representing this family of gates is introduced. In Section 5, a fuzzy type representation for Fredkin gates based on Product and Łukasiewicz t-norms is given.

2 Basic Notions

In quantum computation, information is stored and processed by means of quantum systems. A *quantum bit* or *qubit* (state), the fundamental concept of quantum computation, is a pure state that is described by unit vectors in the Hilbert space $\mathbb{C}^2$. In the *standard orthonormal basis* $\{|0\rangle, |1\rangle\}$ of $\mathbb{C}^2$—generally known as the *quantum computational basis* – , $|1\rangle$ is related to the logical truth value and $|0\rangle$ to the logical falsity. The pure states $|\psi\rangle$ in $\mathbb{C}^2$ are representable as superpositions of the basis vectors with complex coefficients ($|\psi\rangle = c_0|0\rangle + c_1|1\rangle$ where $|c_0|^2 + |c_1|^2 = 1$). Further, in the usual representation of quantum computational processes, a quantum circuit is identified with an appropriate composition of quantum gates, mathematically represented by unitary operators acting on pure states of $\otimes^n \mathbb{C}^2$.

In general, however, a real quantum system is hardly found in a pure state. This lack of purity may be caused, for example, by the inefficiency of the preparation procedure, or as well by the environment-induced decoherence. Further, there are also important quantum physical processes that can not be encoded by unitary evolutions. For example, towards the end of computation, usually, a non-unitary operation—namely a measurement—is applied, as a consequence the final state of the system becomes a probability distribution over pure states, i.e. a *mixed state*. In light of these facts, several authors have discussed a more general model of quantum computational processes wherein the *pure states* are replaced by *mixed states*, and the operations are modeled as *quantum-processes* [4, 5]. In what follows, we give a short description of this mathematical model.

To each vector of the quantum computational basis of $\mathbb{C}^2$ we associate two density operators $P_0 = |0\rangle\langle 0|$ and $P_1 = |1\rangle\langle 1|$ forming the standard computational basis in this framework. They are generalized into n-dimensional systems as $P_1^{(n)} \equiv (\otimes^{(n-1)} I) \otimes P_1$, defined over the Hilbert space of $\otimes^n \mathbb{C}^2$, wherein, I is the 2×2 identity matrix. Clearly, $P_1^{(n)}$ is a 2^n-square matrix. Analogously, $P_0^{(n)}$ is defined as $P_0^{(n)} \equiv (\otimes^{(n-1)} I) \otimes (I - P_1) = I^{(n)} - P_1^{(n)}$.

Using the well known Born rule and referring to the standard approach of the quantum computational logic [12, 14, 19], we introduce the truth-probability value of a density operator as:

$$p(\rho_n) = \operatorname{tr}\left[P_1^{(n)} \rho_n\right].$$

In this powerful framework, the role of quantum gates is assumed by *quantum operations* [17, 21]. A *quantum operation* is representable by a linear operator $\mathcal{E} : \mathcal{L}(H_1) \to \mathcal{L}(H_2)$, where, $\mathcal{L}(H_i)$ is the space of linear operators on the complex Hilbert space H_i $(i = 1, 2)$, representable as $\mathcal{E}(\rho) = \sum_i A_i \rho A_i^\dagger$ where A_i are operators satisfying $\sum_i A_i^\dagger A_i = I$ (Kraus representation [17]). It can be seen that a quantum operation maps density operators into density operators. For any unitary operator U, the quantum operation $U\rho U^\dagger$ is naturally associated. This model is known as "*quantum computation with mixed states*" [1, 13, 14, 19] and it allows us to represent quantum physical processes including irreversible processes such as measurement in the middle or towards the end of the computation.

3 Fredkin Gate

The Fredkin gate, introduced by Edward Fredkin [10], is a *universal reversible classical logic gate*, meaning that, any classical reversible circuit can be built from an ensemble of Fredkin gates alone.

This gate has three input bits and three output bits. The first bit, say x, taken to be the control bit, remains unaffected by the action of the gate. The second and the third bits, say y and z, are taken to be the target bits that are to be swapped if and only if the control bit x is set to 1, and kept unchanged otherwise. In other words, the Fredkin gate can be understood as a *control-swap gate*.

Definition 3.1 The application of the Fredkin gate to a set of three bits $\{x, y, z\}$ is dictated by:

$$F(x, y, z) = (x,\;\; y \widehat{+} x(y \widehat{+} z),\;\; z \widehat{+} x(y \widehat{+} z))$$

where, $\widehat{+}$ is the sum modulo 2.

The Fredkin gate can be used to reproduce the classical AND gate. For example, when $z = 0$, the $F(x, y, 0)$ mimics the classical AND between x, y. Similarly, when z is set to 1, the $F(x, y, 1)$ mimics the classical disjunction between x and y. Now, the Fredkin gates can also be naturally extended as a quantum gate in the following way.

Definition 3.2 Let $|x\rangle = |x_1, x_2, \ldots, x_n\rangle$, $|y\rangle = |y_1, y_2, \ldots, y_m\rangle$ and $|z\rangle = |z_1, z_2 \ldots, z_l\rangle$ be vectors of the standard orthonormal basis in $\otimes^n \mathbb{C}^2$, $\otimes^m \mathbb{C}^2$ and $\otimes^l \mathbb{C}^2$, respectively. Then, the quantum Fredkin gate is defined by the following equation:

$$F^{(n,m,l)}|x, y, z\rangle = |x\rangle|y_1 \ldots y_{m-1}, y_m \widehat{+} x_n(y_m \widehat{+} z_l)\rangle|z_1 \ldots z_{l-1}, z_l \widehat{+} x_n(y_m \widehat{+} z_l)\rangle.$$

A crucial feature of the classical Fredkin gate, in contrast to other classical universal gates—for example the Toffoli gate -, is that the Fredkin gate is *logically conservative*. This means to say that the number of 1's present in the output of the gate is the same as the number of $1'$s in its input. In other words, the *parity of bits* remains unchanged during the operation of logically-conservative gates like the Fredkin Gate [3, 22, 23].

This aspect turns out to be advantageous for building computational circuits that could dissipate less energy (in comparison to the circuits of non-conservative gates) during their operational cycles [3, 22]. This is in the light of the well known *Landauer's principle*, according to which, *there is an unavoidable heat-dissipation-cost associated with every bit of information that gets erased*. The theoretical lower bound to the heat-generation of this type is argued to be $K_B T \;\log 2$. This link between the thermodynamical reversibility and the logical conservativity is due tothe well known *statistical-inviolability* of the second law of thermodynamics [2, 3, 18, 22] following a deconstruction of the much debated *Maxwell's demon*.

This crucial aspect of the classical conservative gates gets extended to the quantum gates as follows. Firstly, at a design level, if a gate-module in a given circuit is logically irreversible (meaning that the information encoded by the input states is not entirely recoverable by using the output states alone), then it must be the case that some information about the input states is lost from the gate-module in question. But, the problem of weather this information is irreversibly lost or not, depends on the details of the physical implementation of the gate: this information may either be irreversibly lost—resulting in heat-dissipation, or be just *hidden away* (in a *deterministically retrievable* manner) in some other module of the physical circuit. In a similar case, it may not result in a heat generation, but perhaps costing a memory-resource overhead. The *logically reversible gates*, like the Toffoli and the Fredkin gate, would naturally avoid this type of dissipation at the very design-level itself by leaving a one-to-one correspondence between the output and the input, thereby keeping all the information about the inputs within the same specific gate-module.

However, at the level of physical implementation, there is a further possibility that the operational cycles of even a reversible gate would involve an erasure of some bits of information. There are several factors which could contribute to this information-erasure resulting in dissipation. The main possible reason for this is as follows: though, in theory, all the memory-states are ideally expected to be equally probable, at any non-zero temperature, the memory states of a physical device would be unequally populated following a Boltzman-distribution. This is especially the case of those quantum systems in which the encoding is done onto the energy states of a quantum system. Therein, in ambient temperatures, the ground state is highly populated and the exited states are less populated, following a Boltzman distribution. Further, there is also a natural loss of population from the exited states (also called as the spontaneous emission). This would make it necessary that a standard repumping mechanism be incorporated to retain the memory-states that are encoded using the excited states. These factors would summarily result in an asymmetry in the operational (thermo-economical) cost of different memory-states belonging to the same physical system.

However, there is a possible way of circumventing the above type of dissipation by using those family gates which are not only *logically-reversible* but also *logically-conservative*, like e.g., the Fredkin gate. The strategy is to use an encoding of information such that the most recurring bits of an input is mapped to the most-stable states respectively. Then, the conservativity of the Fredkin gate would guarantee that the number of excited states remains unaltered throughout the operational cycles of the gate: the output would have the same number of excited states as the input was, and hence no extra stabilization cost is required.

Thus, following the above arguments, even though all the quantum gates by construction are reversible, it becomes desirable to design the circuits based on the logically conservative gates like the Fredkin gate that is characterized in the present work.

4 Operational Representation of the Quantum Fredkin Gate

In order to provide an operationally useful matrix representation of the quantum Fredkin gate, we first need to introduce a generalization of the *SWAP* gate.

Definition 4.1 Let $|x\rangle = |x_1, x_2, \ldots, x_m\rangle$ and $|y\rangle = |y_1, y_2, \ldots, y_l\rangle$ be vectors of the standard orthonormal basis in $\otimes^m \mathbb{C}^2$ and $\otimes^l \mathbb{C}^2$, respectively. Then, the quantum gate $\mathrm{SWAP}^{(m,l)}$ is defined as:

$$SWAP^{(m,l)}|y_1, \ldots, y_m\rangle|z_1, \ldots, z_l\rangle = |y_1, \ldots, y_{m-1}, z_l\rangle|z_1, \ldots, z_{l-1}, y_m\rangle.$$

The following proposition describes the matrix representation of $SWAP^{(m,l)}$.

Proposition 4.1 1. *The matrix form of $SWAP^{(m,l)}$ is provided by the following formula:*

$$SWAP^{(m\,,\,l)} = I^{(m-1)} \otimes \left[\begin{array}{c|c} P_0^{(l)} & L_1^{(l)} \\ \hline L_0^{(l)} & P_1^{(l)} \end{array}\right]$$
$$= P_0^{(m)} \otimes P_0^{(l)} + P_1^{(m)} \otimes P_1^{(l)} + L_0^{(m)} \otimes L_1^{(l)} + L_1^{(m)} \otimes L_0^{(l)}$$

where $L_0^{(l)}$ and $L_1^{(l)}$ are defined as $L_0^{(l)} = I^{(l-1)} \otimes L_0$ and $L_1^{(l)} = I^{(l-1)} \otimes L_1$ respectively, where $L_0 = |0\rangle\langle 1|$ and $L_1 = |1\rangle\langle 0|$ are also known as Ladder operators;[1]

2. *The $SWAP^{(m,l)}$ operator is unitary.*

Proof 1. Let $|x\rangle = |x_1, \ldots, x_n\rangle$, $|y\rangle = |y_1, \ldots, y_m\rangle$ and $|z\rangle = |z_1, \ldots, z_l\rangle$ be the basis vectors in $\otimes^n \mathbb{C}^2$, $\otimes^m \mathbb{C}^2$ and $\otimes^l \mathbb{C}^2$, where $|y_m\rangle$ and $|z_l\rangle$ are given by: $|y_m\rangle \equiv \begin{pmatrix} y_{m_0} \\ y_{m_1} \end{pmatrix}$; *and* $|z_l\rangle \equiv \begin{pmatrix} z_{l_0} \\ z_{l_1} \end{pmatrix}$.

It is straightforward to see that $SWAP^{(m\,,\,l)} = I^{(m-1)} \otimes SWAP^{(1\,,\,l)}$. Then, we confine our proof to the simplest case of $SWAP^{(1,l)}$.

Indeed, let us consider the product $SWAP^{(1\,,\,l)}(|y_m\rangle, |z_1, \ldots, z_l\rangle)$

$$= \left[\begin{array}{c|c} P_0^{(l)} & L_1^{(l)} \\ \hline L_0^{(l)} & P_1^{(l)} \end{array}\right] \cdot \begin{pmatrix} y_{m_0} \\ y_{m_1} \end{pmatrix} \otimes |z_1, \ldots, z_l - 1\rangle \otimes \begin{pmatrix} z_{l_0} \\ z_{l_1} \end{pmatrix}$$

$$= \left[\begin{array}{c|c} P_0^{(l)} & L_1^{(l)} \\ \hline L_0^{(l)} & P_1^{(l)} \end{array}\right] \cdot \left(\begin{array}{c} y_{m_0}|z_1, \ldots, z_l - 1\rangle \otimes \begin{pmatrix} z_{l_0} \\ z_{l_1} \end{pmatrix} \\ \hline y_{m_1}|z_1, \ldots, z_l - 1\rangle \otimes \begin{pmatrix} z_{l_0} \\ z_{l_1} \end{pmatrix} \end{array}\right)$$

$$= \left(\begin{array}{c} P_0^{(l)} \cdot y_{m_0}|z_1, \ldots, z_l - 1\rangle \otimes \begin{pmatrix} z_{l_0} \\ z_{l_1} \end{pmatrix} + L_1^{(l)} \cdot y_{m_1}|z_1, \ldots, z_l - 1\rangle \otimes \begin{pmatrix} z_{l_0} \\ z_{l_1} \end{pmatrix} \\ \hline L_0^{(l)} \cdot y_{m_0}|z_1, \ldots, z_l - 1\rangle \otimes \begin{pmatrix} z_{l_0} \\ z_{l_1} \end{pmatrix} + P_1^{(l)} \cdot y_{m_1}|z_1, \ldots, z_l - 1\rangle \otimes \begin{pmatrix} z_{l_0} \\ z_{l_1} \end{pmatrix} \end{array}\right)$$

$$= \left(\begin{array}{c} I^{(l-1)} \cdot y_{m_0}|z_1, \ldots, z_l - 1\rangle \otimes P_0 \begin{pmatrix} z_{l_0} \\ z_{l_1} \end{pmatrix} + I^{(l-1)} \cdot y_{m_1}|z_1, \ldots, z_l - 1\rangle \otimes L_1 \begin{pmatrix} z_{l_0} \\ z_{l_1} \end{pmatrix} \\ \hline I^{(l-1)} \cdot y_{m_0}|z_1, \ldots, z_l - 1\rangle \otimes L_0 \begin{pmatrix} z_{l_0} \\ z_{l_1} \end{pmatrix} + I^{(l-1)} \cdot y_{m_1}|z_1, \ldots, z_l - 1\rangle \otimes P_1 \begin{pmatrix} z_{l_0} \\ z_{l_1} \end{pmatrix} \end{array}\right)$$

$$= \left(\begin{array}{c} \begin{pmatrix} y_{m_0} z_{l_0}|z_1, \ldots, z_l - 1\rangle \\ y_{m_1} z_{l_0}|z_1, \ldots, z_l - 1\rangle \end{pmatrix} \\ \hline \begin{pmatrix} y_{m_0} z_{l_1}|z_1, \ldots, z_l - 1\rangle \\ y_{m_1} z_{l_1}|z_1, \ldots, z_l - 1\rangle \end{pmatrix} \end{array}\right)$$

$$= \begin{pmatrix} z_{l_0} \\ z_{l_1} \end{pmatrix} \otimes |z_1, \ldots, z_l - 1\rangle \otimes \begin{pmatrix} y_{m_0} \\ y_{m_1} \end{pmatrix}$$

$$= |z_l\rangle \otimes |z_1, \ldots, z_l - 1\rangle \otimes |y_m\rangle.$$

[1] The ladder operators provided in the above construction of the $SWAP^{(m,l)}$ gate are used to represent the multilevel atomic systems like *e.g.*, Rubidium and other Rydberg atoms which are well regarded as promising candidates for implementations in quantum computing.

Let us notice that the last qubits are swapped in the output with respect to the input, as required by Definition 4.1. Further,

$$\begin{aligned}
SWAP^{(m,l)} &= I^{(m-1)} \otimes \left[\begin{array}{c|c} P_0^{(l)} & L_1^{(l)} \\ \hline L_0^{(l)} & P_1^{(l)} \end{array}\right] \\
&= I^{(m-1)} \otimes \left(\left[\begin{array}{c|c} I^{(l-1)} \otimes P_0 & 0 \\ \hline 0 & 0 \end{array}\right] + \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & I^{(l-1)} \otimes P_1 \end{array}\right] \right. \\
&\qquad \left. + \left[\begin{array}{c|c} 0 & I^{(l-1)} \otimes L_1 \\ \hline 0 & 0 \end{array}\right] + \left[\begin{array}{c|c} 0 & 0 \\ \hline I^{(l-1)} \otimes L_0 & 0 \end{array}\right] \right) \\
&= I^{(m-1)} \otimes (P_0 \otimes I^{(l-1)} \otimes P_0 + P_1 \otimes I^{(l-1)} \otimes P_1 + L_0 \otimes I^{(l-1)} \otimes L_1 \\
&\quad + L_1 \otimes I^{(l-1)} \otimes L_0).
\end{aligned}$$

Hence, our claim.

2. $(SWAP^{(m,l)})^\dagger = I^{(m-1)} \otimes \left[\begin{array}{c|c} (P_0^{(l)})^\dagger & (L_0^{(l)})^\dagger \\ \hline (L_1^{(l)})^\dagger & (P_1^{(l)})^\dagger \end{array}\right]$ but, reminding that $P_0^\dagger = P_0$, $P_1^\dagger = P_1$, $L_0^\dagger = L_1$ and $L_1^\dagger = L_0$, follows that $(SWAP^{(m,l)})^\dagger = (SWAP^{(m,l)})$.

   But, by definition, $(SWAP^{(m,l)}) \cdot (SWAP^{(m,l)}) = I^{(m+l)}$. Hence, our claim.

□

Proposition 4.2 *For any natural number* $n, m, l \geq 1$, *the Fredkin gate* $F^{(n,m,l)}$ *assumes the following matrix representation:*

$$F^{(n,m,l)} = P_0^{(n)} \otimes I^{(m+l)} + P_1^{(n)} \otimes SWAP^{(m,l)}.$$

Proof By following the standard construction of controlled operators (see, e.g., section 4.3 in [21]), if $U^{(m)}$ is a unitary m-qubit gate, then the controlled-U gate operating on $m+1$ qubits assumes the following block-representation:

$$Control - U^{(m)} = \left[\begin{array}{c|c} I^{(m)} & 0 \\ \hline 0 & U^{(m)} \end{array}\right].$$

This block representation allows us to end up with the following operational representation of an arbitrary $Control - U^{(m)}$ gate:

$$Control - U^{(m)} = \left[\begin{array}{c|c} I^{(m)} & 0 \\ \hline 0 & U^{(m)} \end{array}\right] = \left[\begin{array}{c|c} I^{(m)} & 0 \\ \hline 0 & 0 \end{array}\right] + \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & U^{(m)} \end{array}\right] = P_0 \otimes I^{(m)} + P_1 \otimes U^{(m)}.$$

Further, for the case when the control/ancilla is a n-dimensional qubit, then the generalized n-control unitary $Control - U^{(n,m)}$ is readily verifiable to be given by:

$$\begin{aligned}
Control - U^{(n,m)} &= I^{(n-1)} \otimes \left[\begin{array}{c|c} I^{(m)} & 0 \\ \hline 0 & U^{(m)} \end{array}\right] = P_0^{(n)} \otimes I^{(m)} + P_1^{(n)} \otimes U^{(m)} \\
&= I^{(n-1)} \otimes \left(P_0 \otimes I^{(m)} + P_1 \otimes U^{(m)} \right).
\end{aligned}$$

Now, let us recall that, by Definition 3.2 and as described in the previous Section, the Fredkin gate can be considered as a *control-swap gate*. Therefore:

$$F^{(n,m,l)} = I^{(n-1)} \otimes \left[\begin{array}{c|c} I^{(m+l)} & 0 \\ \hline 0 & SWAP^{(m,l)} \end{array}\right] = I^{(n-1)} \otimes \left(P_0 \otimes I^{(m)} + P_1 \otimes SWAP^{(m,l)}\right).$$

Further, recalling that from the block representation of $SWAP^{(m,l)}$ easily turns out the unitarity of the $SWAP$ gate, by very similar considerations also the unitarity of $F^{(n,m,l)}$ can be proven. □

The operational and the matrix representations of the Fredkin gate turn out to be very useful to achieve easy computations of several algebraic properties of this gate. In the next Section we provide a calculation to evaluate how the probability of a factorized density operator changes after the application of the Fredkin gate.

5 A Fuzzy Representation of the Fredkin Gate

Let us recall that, for an arbitrary quantum operator A and an arbitrary density operator ρ of same dimension (n), the *truth-probability* of ρ after the application of A is defined by:

$$p[\mathbb{A}^{(n)}, \rho] = Tr[P_1^{(n)} \cdot (A \cdot \rho \cdot A^\dagger)] = Tr[(A^\dagger \cdot P_1^{(n)} \cdot A) \cdot \rho].$$

Proposition 5.1 *The probability of an arbitrary product density operator $\rho = \rho_n \otimes \rho_m \otimes \rho_l$ after the application of the Fredkin gate, is given by:*

$$p[\mathbb{F}^{(n,m,l)}, \rho] = (1 - p(\rho_n))\ p(\rho_l) + p(\rho_n)\ p(\rho_m).$$

Proof

$$\begin{aligned} p[\mathbb{F}^{(n,m,l)}, \rho] &= Tr\left[P_1^{(n+m+l)} \cdot \left(F^{(n,m,l)} \cdot \rho \cdot F^{(n,m,l)}\right)\right] \\ &= Tr\left[\left(F^{(n,m,l)} \cdot P_1^{(n+m+l)} \cdot F^{(n,m,l)}\right)\rho\right]. \end{aligned}$$

By using the operational representation of $F^{(n,m,l)}$ detailed in the previous Section, we obtain:

$$\begin{aligned} & F^{(n,m,l)} \cdot P_1^{(n+m+l)} \cdot F^{(n,m,l)} \\ &= I^{(n-1)} \otimes \Big[\left(P_0 \otimes I^{(m+l)} + P_1 \otimes SWAP^{(m,l)}\right) \cdot \left(I^{(m+l)} \otimes P_1\right) \\ & \quad \cdot \left(P_0 \otimes I^{(m+l)} + P_1 \otimes SWAP^{(m,l)}\right)\Big] \\ &= I^{(n-1)} \otimes \begin{bmatrix} ((P_0 \otimes I^{(m+l)}) \cdot (I^{(m+l)} \otimes P_1) \cdot (P_0 \otimes I^{(m+l)})) \\ +(P_0 \cdot I \cdot P_1) \otimes (\ldots) \\ +(P_1 \cdot I \cdot P_0) \otimes (\ldots) \\ +(P_1 \cdot I \cdot P_1) \otimes (I^{(m-1)} \cdot I^{(m-1)} \cdot I^{(m-1)}) \otimes (SWAP^{(1,l)} \cdot (I \otimes P_1^{(l)}) \cdot SWAP^{(1,l)}) \end{bmatrix} \end{aligned}$$

Let us recall that, $P_0 \cdot I \cdot P_1 = P_1 \cdot I \cdot P_0$ that correspond to the null matrix 0. Further, $SWAP^{(m,l)} = I^{(m-1)} \otimes SWAP^{(1,l)}$.

$$= I^{(n-1)} \otimes \begin{bmatrix} (P_0 \cdot I \cdot P_0) \otimes (I^{(m+l-1)} \cdot P_1^{(m+l-1)} \cdot I^{(m+l-1)}) \\ +0 \\ +0 \\ +(P_1 \cdot I \cdot P_1) \otimes (I^{(m-1)} \cdot I^{(m-1)} \cdot I^{(m-1)}) \otimes (SWAP^{(1,l)} \cdot (I \otimes P_1^{(l)}) \cdot SWAP^{(1,l)}) \end{bmatrix}$$

Let us recall that, for density matrices A,B,C,D of appropriate dimension, is $SWAP^{(m,l)} \cdot \big(A^{(m-1)} \otimes B \otimes C^{(l-1)} \otimes D\big) \cdot SWAP^{(m,l)} = A^{(m-1)} \otimes D \otimes C^{(l-1)} \otimes B$. Hence,

$$
\begin{aligned}
&= I^{(n-1)} \otimes \left[P_0 \otimes P_1^{(m+l)} + P_1 \otimes I^{(m-1)} \otimes P_1 \otimes I^{(l)}\right] \\
&= P_0^{(n)} \otimes I^{(m)} \otimes P_1^{(l)} + P_1^{(n)} \otimes P_1^{(m)} \otimes I^{(l)} \\
&= \left(I^{(n)} - P_1^{(n)}\right) \otimes I^{(m)} \otimes P_1^{(l)} + P_1^{(n)} \otimes P_1^{(m)} \otimes I^{(l)}.
\end{aligned}
$$

Therefore, for the truth-probability of the Fredkin operation on a product input state of the form $\rho \equiv \rho_n \otimes \rho_m \otimes \rho_l$, we obtain:

$$p[\mathbb{F}^{(n,m,l)}, \rho] = Tr\left[\left(\left(I^{(n)} - P_1^{(n)}\right) \otimes I^{(m)} \otimes P_1^{(l)} + P_1^{(n)} \otimes P_1^{(m)} \otimes I^{(l)}\right) \cdot (\rho_n \otimes \rho_m \otimes \rho_l)\right]$$

which can be reduced in a straightforward manner to $(1 - p(\rho_n))\ p(\rho_l) + p(\rho_n)\ p(\rho_m)$.

□

As a first remark, let us notice that, since $0 \leq p[\mathbb{F}^{(n,m,l)}, \rho_n \otimes \rho_m \otimes \rho_l] \leq 1$, the above sum is a Łukasiewicz sum that can be written as:

$$p[\mathbb{F}^{(n,m,l)}, \rho_n \otimes \rho_m \otimes \rho_l] = \neg p(\rho_n) \cdot p(\rho_l) \oplus p(\rho_n) \cdot p(\rho_m).$$

Finally, in the special case where the third bit ρ_l corresponds to the ancilla P_0, then the probability of the Fredkin gate behaves as a product:

$$p[\mathbb{F}^{(n,m,1)}, \rho_n \otimes \rho_m \otimes P_0] = p(\rho_n) \cdot p(\rho_m).$$

Let us notice that the last two expressions are polynomial terms interpreted in the standard product MV-algebra defined in the real interval [0, 1]. It defines a fuzzy representation of the Fredkin gate as a polynomial term in the language of PMV-algebras.

6 Conclusion

In this work we have provided a calculation about the probability value after the application of the Fredkin gate to a product state $\rho_n \otimes \rho_m \otimes \rho_l$. In this way we are able to provide a fuzzy-type representation of the Fredkin gate in terms of a polynomial in the language of PMV-algebra. Nevertheless, the operational and the matricial expressions of the Fredkin gate obtained above will be also useful to study the behavior of the gate in the more general case of non-product states.

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