L'ELEMENTO PIASTRA FLESSIONALE A 4 NODI DI MELOSH

La teoria di Kirchhoff è applicabile alle piastre sottili nelle quali lo scorrimento trasversale è trascurabile. L'energia elastica nella piastra dipende dalle deformazioni nel piano ε_x, ε_y e γ_{xy} che dipendono dallo spostamento trasversale $w = w(x,y)$, come mostrato dalle seguenti equazioni:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = -z \begin{Bmatrix} w_{,xx} \\ w_{,yy} \\ 2w_{,xy} \end{Bmatrix} = -z \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}$$

Il legame sforzi-deformazioni è il seguente:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{-Ez}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}$$

e poiché il legame sforzi – momenti flettenti unitari è il seguente:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{12z}{t^3} \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix}$$

risulta che il legame "*momenti flettenti unitari – curvature*" è il seguente:

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \frac{-Et^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} = -[\boldsymbol{D_K}]\{\boldsymbol{\kappa}\}$$

dove

$$[\boldsymbol{D_K}] = \frac{Et^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \quad \text{e} \quad \{\boldsymbol{\kappa}\} = \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}$$

Il punto di partenza per la formulazione della matrice di rigidezza elementare è l'energia elastica:

$$U_e = \int_{vol} \frac{1}{2} \{\boldsymbol{\sigma}\}^T \{\boldsymbol{\varepsilon}\} \cdot dVol$$

dove, come appena visto:

$$\{\boldsymbol{\sigma}\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{-Ez}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \quad \text{e} \quad \{\boldsymbol{\varepsilon}\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = -z \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} = -z\{\boldsymbol{\kappa}\}$$

da cui:

$$U_e = \int_{vol} \frac{1}{2} \{\boldsymbol{\sigma}\}^T \{\boldsymbol{\varepsilon}\} \cdot dVol = \frac{1}{2} \int_{vol} \{\boldsymbol{\kappa}\}^T \frac{Ez^2}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \{\boldsymbol{\kappa}\} \cdot dVol$$

Posto $dVol = dz \cdot dA$ e $dA = dx \cdot dy$ che rappresenta un'area infinitesima appartenente al piano medio, l'integrazione lungo lo spessore t fornisce:

$$U_e = \frac{1}{2} \int_A \{\boldsymbol{\kappa}\}^T \frac{Et^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \{\boldsymbol{\kappa}\} \cdot dVol = \int_A \frac{1}{2} \{\boldsymbol{\kappa}\}^T [\boldsymbol{D_K}] \{\boldsymbol{\kappa}\} \cdot dA$$

Individuata una funzione d'interpolazione dello spostamento w in funzione dei gradi di libertà nodali $\{\boldsymbol{d}\}$, è possibile differenziarla per ottenere le curvature $\{\boldsymbol{\kappa}\}$. Per un elemento a NNe nodi abbiamo:

$$w = \underbrace{[\boldsymbol{N}]}_{1\times 3NNe} \{\boldsymbol{d}\} \qquad \text{quindi} \qquad \{\boldsymbol{\kappa}\} = \underbrace{[\boldsymbol{B}]}_{3\times 3NNe} \{\boldsymbol{d}\}$$

I gradi di libertà di un elemento di Kirchhoff sono:

$$\{\boldsymbol{d}\} = \{w_1 \quad w_{,x1} \quad w_{,y1} \quad \cdots \quad w_N \quad w_{,xN} \quad w_{,yN}\}^T$$

da cui l'energia elastica risulta:

$$U_e = \int_A \frac{1}{2}\{\boldsymbol{\kappa}\}^T[\boldsymbol{D_K}]\{\boldsymbol{\kappa}\}dA = \frac{1}{2}\{\boldsymbol{d}\}^T \int_A [\boldsymbol{B}]^T[\boldsymbol{D_K}][\boldsymbol{B}]dA \cdot \{\boldsymbol{d}\} = \frac{1}{2}\{\boldsymbol{d}\}^T[\boldsymbol{k}]\{\boldsymbol{d}\}$$

dove la matrice di rigidezza dell'elemento risulta:

$$\underbrace{[\boldsymbol{k}]}_{3NNe\times 3NNe} = \int_A [\boldsymbol{B}]^T[\boldsymbol{D_K}][\boldsymbol{B}]\, dA$$

Nel 1963 Melosh ha proposto un elemento che si basa su un polinomio contenente 12 coefficienti:

$$w(\xi,\eta) = a_1 + a_2\xi + a_3\eta + a_4\xi^2 + a_5\xi\eta + a_6\eta^2 + a_7\xi^3 + a_8\xi^2\eta + a_9\xi\eta^2 + a_{10}\eta^3 + a_{11}\xi^3\eta + a_{12}\xi\eta^3$$

o, in forma matriciale:

$$w = \{1 \quad \xi \quad \eta \quad \xi^2 \quad \xi\eta \quad \eta^2 \quad \xi^3 \quad \xi^2\eta \quad \xi\eta^2 \quad \eta^3 \quad \xi^3\eta \quad \xi\eta^3\}\{\boldsymbol{a}\}$$

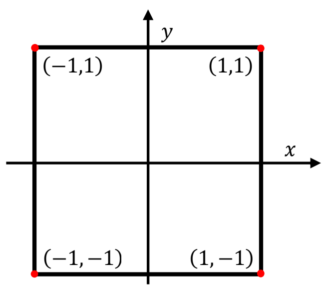
Si tratta di un polinomio incompleto di quarto grado in quanto mancano i coefficienti che moltiplicano ξ^4, $\xi^2\eta^2$ e η^4. Il vettore $\{\boldsymbol{a}\}$ contiene 12 coordinate generalizzate che bisogna scambiare con i 12 gradi di libertà nodali $\{\boldsymbol{d}\}$ con il metodo già descritto nei capitoli precedenti. E' quindi possibile calcolare la matrice $[\boldsymbol{B}]$ e la matrice di rigidezza elementare $[\boldsymbol{k}]$. La convergenza dell'elemento a 12 coefficienti non è monotona, quindi una mesh costituita da questi elementi può essere troppo rigida in alcuni problemi e troppo flessibile in altri. Ciò capita perché l'inclinazione della normale al piano medio, per esempio $\partial w/\partial\xi$ lungo il lato $\xi = costante$, non è compatibile tra elementi contigui.

Dimostrazione

Per semplicità poniamo che l'elemento sia allineato alle coordinate globali x e y come riportato nella figura a lato, quindi $x \equiv \xi$ e $y \equiv \eta$.

E' sufficiente seguire la procedura già utilizzata nei capitoli precedenti esprimendo lo spostamento w in forma matriciale:

$$w = \{1 \quad x \quad y \quad x^2 \quad xy \quad y^2 \quad x^3 \quad x^2y \quad xy^2 \quad y^3 \quad x^3y \quad xy^3\}\{a\}$$



Le variabili nodali sono w, $\frac{\partial w}{\partial x} = w_{,x}$ e $\frac{\partial w}{\partial y} = w_{,y}$ per cui:

$$\begin{Bmatrix} w \\ w_{,x} \\ w_{,y} \end{Bmatrix} = \begin{bmatrix} 1 & x & y & x^2 & xy & y^2 & x^3 & x^2y & xy^2 & y^3 & x^3y & xy^3 \\ 0 & 1 & 0 & 2x & y & 0 & 3x^2 & 2xy & y^2 & 0 & 3x^2y & y^3 \\ 0 & 0 & 1 & 0 & x & 2y & 0 & x^2 & 2xy & 3y^2 & x^3 & 3xy^2 \end{bmatrix}\{a\}$$

Sostituendo i valori delle coordinate nodali possiamo scrivere il seguente sistema e trovare il vettore dei parametri $\{a\}$:

$$\begin{Bmatrix} \begin{Bmatrix} w \\ w_{,x} \\ w_{,y} \end{Bmatrix}_1 \\ \begin{Bmatrix} w \\ w_{,x} \\ w_{,y} \end{Bmatrix}_2 \\ \begin{Bmatrix} w \\ w_{,x} \\ w_{,y} \end{Bmatrix}_3 \\ \begin{Bmatrix} w \\ w_{,x} \\ w_{,y} \end{Bmatrix}_4 \end{Bmatrix} = \begin{bmatrix} 1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 0 & 1 & 0 & -2 & -1 & 0 & 3 & 2 & 1 & 0 & -3 & -1 \\ 0 & 0 & 1 & 0 & -1 & -2 & 0 & 1 & 2 & 3 & -1 & -3 \\ 1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 2 & -1 & 0 & 3 & -2 & 1 & 0 & -3 & -1 \\ 0 & 0 & 1 & 0 & 1 & -2 & 0 & 1 & -2 & 3 & 1 & 3 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 2 & 1 & 0 & 3 & 2 & 1 & 0 & 3 & 1 \\ 0 & 0 & 1 & 0 & 1 & 2 & 0 & 1 & 2 & 3 & 1 & 3 \\ 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 \\ 0 & 1 & 0 & -2 & 1 & 0 & 3 & -2 & 1 & 0 & 3 & 1 \\ 0 & 0 & 1 & 0 & -1 & 2 & 0 & 1 & -2 & 3 & -1 & -3 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \\ a_{10} \\ a_{11} \\ a_{12} \end{Bmatrix} = [A] \cdot \{a\} = \{d\}$$

da cui:

$$\{a\} = [A]^{-1} \cdot \{d\}$$

Le funzioni di forma si trovano eseguendo il prodotto:

$$N = \{1 \quad x \quad y \quad x^2 \quad xy \quad y^2 \quad x^3 \quad x^2y \quad xy^2 \quad y^3 \quad x^3y \quad xy^3\}[A]^{-1}$$

dove:

$$[A]^{-1} = \begin{bmatrix} 2 & 1 & 1 & 2 & -1 & 1 & 2 & -1 & -1 & 2 & 1 & -1 \\ -3 & -1 & -1 & 3 & -1 & 1 & 3 & -1 & -1 & -3 & -1 & 1 \\ -3 & -1 & -1 & -3 & 1 & -1 & 3 & -1 & -1 & 3 & 1 & -1 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 4 & 1 & 1 & -4 & 1 & -1 & 4 & -1 & -1 & -4 & -1 & 1 \\ 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & -1 & 1 & 0 & -1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & 1 & 1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 \\ -1 & -1 & 0 & 1 & -1 & 0 & -1 & 1 & 0 & 1 & 1 & 0 \\ -1 & 0 & -1 & 1 & 0 & 1 & -1 & 0 & 1 & 1 & 0 & -1 \end{bmatrix}$$

e pre moltiplicandola come descritto otteniamo se seguenti funzioni di forma:

$$N_1 = \frac{1}{8} \cdot (2 - 3x - 3y + 4xy + x^3 + y^3 - x^3y - xy^3)$$

$$N_2 = \frac{1}{8} \cdot (1 - x - y - x^2 + xy + x^3 + x^2y - x^3y)$$

$$N_3 = \frac{1}{8} \cdot (1 - x - y + xy - y^2 + xy^2 + y^3 - xy^3)$$

$$N_4 = \frac{1}{8} \cdot (2 + 3x - 3y - 4xy - x^3 + y^3 + x^3y + xy^3)$$

$$N_5 = \frac{1}{8} \cdot (-1 - x + y + x^2 + xy + x^3 - x^2y - x^3y)$$

$$N_6 = \frac{1}{8} \cdot (1 + x - y - xy - y^2 - xy^2 + y^3 + xy^3)$$

$$N_7 = \frac{1}{8} \cdot (2 + 3x + 3y + 4xy - x^3 - y^3 - x^3y - xy^3)$$

$$N_8 = \frac{1}{8} \cdot (-1 - x - y + x^2 - xy + x^3 + x^2y + x^3y)$$

$$N_9 = \frac{1}{8} \cdot (-1 - x - y - xy + y^2 + xy^2 + y^3 + xy^3)$$

$$N_{10} = \frac{1}{8} \cdot (2 - 3x + 3y - 4xy + x^3 - y^3 + x^3y + xy^3)$$

$$N_{11} = \frac{1}{8} \cdot (1 - x + y - x^2 - xy + x^3 - x^2y + x^3y)$$

$$N_{12} = \frac{1}{8} \cdot (-1 + x - y + xy + y^2 - xy^2 + y^3 - xy^3)$$

In conclusione abbiamo:

$$\begin{Bmatrix} w \\ w_{,x} \\ w_{,y} \end{Bmatrix} = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 & N_5 & N_6 & N_7 & N_8 & N_9 & N_{10} & N_{11} & N_{12} \\ \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial x} & \frac{\partial N_5}{\partial x} & \frac{\partial N_6}{\partial x} & \frac{\partial N_7}{\partial x} & \frac{\partial N_8}{\partial x} & \frac{\partial N_9}{\partial x} & \frac{\partial N_{10}}{\partial x} & \frac{\partial N_{11}}{\partial x} & \frac{\partial N_{12}}{\partial x} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} & \frac{\partial N_4}{\partial y} & \frac{\partial N_5}{\partial y} & \frac{\partial N_6}{\partial y} & \frac{\partial N_7}{\partial y} & \frac{\partial N_8}{\partial y} & \frac{\partial N_9}{\partial y} & \frac{\partial N_{10}}{\partial y} & \frac{\partial N_{11}}{\partial y} & \frac{\partial N_{12}}{\partial y} \end{bmatrix} \begin{Bmatrix} \begin{Bmatrix} w \\ w_{,x} \\ w_{,y} \end{Bmatrix}_1 \\ \begin{Bmatrix} w \\ w_{,x} \\ w_{,y} \end{Bmatrix}_2 \\ \begin{Bmatrix} w \\ w_{,x} \\ w_{,y} \end{Bmatrix}_3 \\ \begin{Bmatrix} w \\ w_{,x} \\ w_{,y} \end{Bmatrix}_4 \end{Bmatrix}$$

Le derivate delle funzioni di forma valgono:

$$\frac{\partial N_1}{\partial x}=\frac{(-3+4y+3x^2-3x^2y-y^3)}{8}$$
$$\frac{\partial N_2}{\partial x}=\frac{(-1\ -2x+y+3x^2+2xy-3x^2y)}{8}$$
$$\frac{\partial N_3}{\partial x}=\frac{(-1+y+y^2-y^3)}{8}$$
$$\frac{\partial N_4}{\partial x}=\frac{(3-4y-3x^2+3x^2y+y^3)}{8}$$
$$\frac{\partial N_5}{\partial x}=\frac{(-1+2x+y+3x^2-2xy-3x^2y)}{8}$$
$$\frac{\partial N_6}{\partial x}=\frac{(1-y\ -y^2+y^3)}{8}$$
$$\frac{\partial N_7}{\partial x}=\frac{(3+4y-3x^2-3x^2y-y^3)}{8}$$
$$\frac{\partial N_8}{\partial x}=\frac{(-1+2x-y+3x^2+2xy+3x^2y)}{8}$$
$$\frac{\partial N_9}{\partial x}=\frac{(-1-y+y^2+y^3)}{8}$$
$$\frac{\partial N_{10}}{\partial x}=\frac{(-3-4y+3x^2+3x^2y+y^3)}{8}$$
$$\frac{\partial N_{11}}{\partial x}=\frac{(-1-2x-y+3x^2-2xy+3x^2y)}{8}$$
$$\frac{\partial N_{12}}{\partial x}=\frac{(1\ +y-y^2-y^3)}{8}$$

$$\frac{\partial N_1}{\partial y}=\frac{(-3+4x+3y^2-x^3-3xy^2)}{8}$$
$$\frac{\partial N_2}{\partial y}=\frac{(\ -1+x+x^2-x^3)}{8}$$
$$\frac{\partial N_3}{\partial y}=\frac{(-1+x-2y+2xy+3y^2-3xy^2)}{8}$$
$$\frac{\partial N_4}{\partial y}=\frac{(-3-4x+3y^2+x^3+3xy^2)}{8}$$
$$\frac{\partial N_5}{\partial y}=\frac{(1+x-x^2-x^3)}{8}$$
$$\frac{\partial N_6}{\partial y}=\frac{(-1-x\ -2y-2xy+3y^2+3xy^2)}{8}$$
$$\frac{\partial N_7}{\partial y}=\frac{(3+4x-3y^2-x^3-3xy^2)}{8}$$
$$\frac{\partial N_8}{\partial y}=\frac{(-1-x+x^2+x^3)}{8}$$
$$\frac{\partial N_9}{\partial y}=\frac{(-1-x+2y+2xy+3y^2+3xy^2)}{8}$$
$$\frac{\partial N_{10}}{\partial y}=\frac{(3-4x-3y^2+x^3+3xy^2)}{8}$$
$$\frac{\partial N_{11}}{\partial y}=\frac{(1-x-x^2+x^3)}{8}$$
$$\frac{\partial N_{12}}{\partial y}=\frac{(-1+x+2y-2xy+3y^2-3xy^2)}{8}$$

L'inclinazione della normale al piano medio, per esempio $\partial w/\partial x$ lungo il lato $x=costante$, non è compatibile tra elementi adiacenti.

$$w_{,x}=\sum_{i=1}^{12}\frac{\partial N_i}{\partial x}d_i$$

Perché l'elemento risulti compatibile le variabili di spostamento e rotazione lungo un lato devono dipendere esclusivamente dalle variabili nodali appartenenti allo stesso lato. Per esempio lungo il lato $x=1$ che unisce i nodi n.2 e n.3, le variabili $\begin{Bmatrix} w \\ w_{,x} \\ w_{,y} \end{Bmatrix}$ non devono dipendere da $\begin{Bmatrix} w \\ w_{,x} \\ w_{,y} \end{Bmatrix}_1$ e $\begin{Bmatrix} w \\ w_{,x} \\ w_{,y} \end{Bmatrix}_4$ quindi le funzioni di forma N_1, N_2, N_3, N_{10}, N_{11} e N_{12} valutate in $x=1$ dovrebbero annullarsi.

In realtà lungo la retta $x=1$ abbiamo:

$$N_1(1,y)=\frac{1}{8}\cdot(2-3-3y+4y+1+y^3-y-y^3)=0$$
$$N_2(1,y)=\frac{1}{8}\cdot(1-1\ -y-1+y+1+y-y)=0$$
$$N_3(1,y)=\frac{1}{8}\cdot(1-1-y+y-y^2+y^2+y^3-y^3)=0$$
$$N_{10}(1,y)=\frac{1}{8}\cdot(2-3+3y-4y+1-y^3+y+y^3)=0$$
$$N_{11}(1,y)=\frac{1}{8}\cdot(1-1+y-1-y+1-y+y)=0$$
$$N_{12}(1,y)=\frac{1}{8}\cdot(-1+1\ -y+y+y^2-y^2+y^3-y^3)=0$$

che indica la compatibilità dello spostamento trasversale w. Per quanto riguarda le inclinazioni:

$$\frac{\partial N_1(1,y)}{\partial x}=\frac{1}{8}\cdot(-3+4y+3-3y-y^3)=\frac{y-y^3}{8}$$
$$\frac{\partial N_2(1,y)}{\partial x}=\frac{1}{8}\cdot(-1\ -2+y+3+2y-3y)=0$$
$$\frac{\partial N_3(1,y)}{\partial x}=\frac{1}{8}\cdot(-1+y+y^2-y^3)$$
$$\frac{\partial N_{10}(1,y)}{\partial x}=\frac{1}{8}\cdot(-3-4y+3+3y+y^3)=\frac{(-y+y^3)}{8}$$
$$\frac{\partial N_{11}(1,y)}{\partial x}=\frac{1}{8}\cdot(-1-2-y+3-2y+3y)=0$$
$$\frac{\partial N_{12}(1,y)}{\partial x}=\frac{1}{8}\cdot(1\ +y-y^2-y^3)$$

$$\frac{\partial N_1(1,y)}{\partial y}=\frac{1}{8}\cdot(-3+4+3y^2-1-3y^2)=0$$
$$\frac{\partial N_2(1,y)}{\partial y}=\frac{1}{8}\cdot(-1+1+1-1)=0$$
$$\frac{\partial N_3(1,y)}{\partial y}=\frac{1}{8}\cdot(-1+1-2y+2y+3y^2-3y^2)=0$$
$$\frac{\partial N_{10}(1,y)}{\partial y}=\frac{1}{8}\cdot(3-4-3y^2+1+3y^2)=0$$
$$\frac{\partial N_{11}(1,y)}{\partial y}=\frac{1}{8}\cdot(1-1-1+1)=0$$
$$\frac{\partial N_{12}(1,y)}{\partial y}=\frac{1}{8}\cdot(-1+1+2y-2y+3y^2-3y^2)=0$$

Le seguenti derivate $\frac{\partial N_1(1,y)}{\partial x}$, $\frac{\partial N_3(1,y)}{\partial x}$, $\frac{\partial N_{10}(1,y)}{\partial x}$ e $\frac{\partial N_{12}(1,y)}{\partial x}$ si annullano sono sui nodi 1 e 4 dove $y=\pm 1$, ma lungo il lato che unisce i nodi n.2 e n.3 sono diverse da zero. Ciò dimostra che due elementi che condividono lo stesso lato non sono compatibili perché le rotazioni del lato dipendono dalle rotazioni dei nodi che con sono condivisi dai due elementi.

-- ***Fine Dimostrazione*** --

Riprendiamo la procedura per la formulazione della matrice di rigidezza dell'elemento di Melosh:

$$\underbrace{[\boldsymbol{k}]}_{3NNe\times 3NNe}=\int_A [\boldsymbol{B}]^T[\boldsymbol{D_K}][\boldsymbol{B}]\,dA$$

dove:

$$w=\underbrace{[\boldsymbol{N}]}_{1\times 3NNe}\{\boldsymbol{d}\}\qquad \text{quindi}\qquad \{\boldsymbol{\kappa}\}=\underbrace{[\boldsymbol{B}]}_{3\times 3NNe}\{\boldsymbol{d}\}$$

dove le funzioni di forma $[\boldsymbol{N}]$ sono state sviluppate nelle righe che precedono. La matrice $[\boldsymbol{B}]$ assume la forma seguente:

$$[\boldsymbol{B}]=\begin{bmatrix}\frac{\partial^2 N_1}{\partial x^2} & \frac{\partial^2 N_2}{\partial x^2} & \cdots & \frac{\partial^2 N_{12}}{\partial x^2}\\ \frac{\partial^2 N_1}{\partial y^2} & \frac{\partial^2 N_2}{\partial y^2} & \cdots & \frac{\partial^2 N_{12}}{\partial y^2}\\ \frac{\partial^2 N_1}{\partial x\partial y} & \frac{\partial^2 N_2}{\partial x\partial y} & \cdots & \frac{\partial^2 N_{12}}{\partial x\partial y}\end{bmatrix}$$

L'elemento non è isoparametrico: infatti mentre gli spostamenti si esprimono in funzione di un polinomio incompleto di quarto grado, le coordinate x e y interne all'elemento a 4 nodi si determinano in funzione delle solite funzioni bilineari (dove l'apice "g" è stato aggiunto per distinguere queste funzioni di forma "geometriche" dalle funzioni di forma degli spostamenti):

$$N_1^g=\frac{(1-\xi)(1-\eta)}{4}\quad;\quad N_2^g=\frac{(1-\xi)(1-\eta)}{4}\quad;\quad N_3^g=\frac{(1-\xi)(1-\eta)}{4}\quad;\quad N_4^g=\frac{(1-\xi)(1-\eta)}{4}$$

Poiché le funzioni di forma sono espresse in coordinate naturali ξ ed η, per il calcolo dei coefficienti della matrice $[\boldsymbol{B}]$ è necessario seguire la seguente procedura:

$$\begin{cases}\frac{\partial N_i}{\partial \xi}=\frac{\partial N_i}{\partial x}\frac{\partial x}{\partial \xi}+\frac{\partial N_i}{\partial y}\frac{\partial y}{\partial \xi}\\ \frac{\partial N_i}{\partial \eta}=\frac{\partial N_i}{\partial x}\frac{\partial x}{\partial \eta}+\frac{\partial N_i}{\partial y}\frac{\partial y}{\partial \eta}\end{cases}\qquad \text{o in forma matriciale:}\qquad \begin{Bmatrix}\frac{\partial N_i}{\partial \xi}\\ \frac{\partial N_i}{\partial \eta}\end{Bmatrix}=\begin{bmatrix}\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi}\\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta}\end{bmatrix}\begin{Bmatrix}\frac{\partial N_i}{\partial x}\\ \frac{\partial N_i}{\partial y}\end{Bmatrix}$$

Prima del calcolo della derivata seconda si definiscono le seguenti variabili di x e y:

$$f(\xi,\eta)=\frac{\partial N_i}{\partial \xi}\quad,\quad g(x,y)=\frac{\partial N_i}{\partial x}\qquad \text{e}\qquad h(\xi,\eta)=\frac{\partial N_i}{\partial \eta}\quad,\quad r(x,y)=\frac{\partial N_i}{\partial y}$$

da cui abbiamo:

$$\begin{cases} f(\xi,\eta) = g(x,y)\dfrac{\partial x}{\partial \xi} + r(x,y)\dfrac{\partial y}{\partial \xi} \\ h(\xi,\eta) = g(x,y)\dfrac{\partial x}{\partial \eta} + r(x,y)\dfrac{\partial y}{\partial \eta} \end{cases}$$

Derivando le due espressioni si ottiene:

$$\begin{cases} \dfrac{\partial f(\xi,\eta)}{\partial \xi} = \dfrac{\partial g(x,y)}{\partial \xi}\dfrac{\partial x}{\partial \xi} + g(x,y)\dfrac{\partial^2 x}{\partial \xi^2} + \dfrac{\partial r(x,y)}{\partial \xi}\dfrac{\partial y}{\partial \xi} + r(x,y)\dfrac{\partial^2 y}{\partial \xi^2} \\ \dfrac{\partial h(\xi,\eta)}{\partial \eta} = \dfrac{\partial g(x,y)}{\partial \eta}\dfrac{\partial x}{\partial \eta} + g(x,y)\dfrac{\partial^2 x}{\partial \eta^2} + \dfrac{\partial r(x,y)}{\partial \eta}\dfrac{\partial y}{\partial \eta} + r(x,y)\dfrac{\partial^2 y}{\partial \eta^2} \\ \dfrac{\partial f(\xi,\eta)}{\partial \eta} = \dfrac{\partial g(x,y)}{\partial \eta}\dfrac{\partial x}{\partial \xi} + g(x,y)\dfrac{\partial^2 x}{\partial \xi \partial \eta} + \dfrac{\partial r(x,y)}{\partial \eta}\dfrac{\partial y}{\partial \xi} + r(x,y)\dfrac{\partial^2 y}{\partial \xi \partial \eta} = \dfrac{\partial h(\xi,\eta)}{\partial \xi} \end{cases}$$

Poiché le funzioni di forma geometriche sono bilineari, le loro derivate seconde $\frac{\partial^2 N_i}{\partial \xi^2}$, $\frac{\partial^2 N_i}{\partial \eta^2}$, $\frac{\partial^2 N_i}{\partial \xi^2}$ e $\frac{\partial^2 N_i}{\partial \eta^2}$ sono nulle e di conseguenza anche le $\frac{\partial^2 x}{\partial \xi^2}$, $\frac{\partial^2 x}{\partial \eta^2}$, $\frac{\partial^2 y}{\partial \xi^2}$ e $\frac{\partial^2 y}{\partial \eta^2}$ si annullano. Le derivate seconde miste delle funzioni di forma geometriche valgono:

$$\frac{\partial^2 N_1}{\partial \xi \partial \eta} = \frac{1}{4} \quad ; \quad \frac{\partial^2 N_2}{\partial \xi \partial \eta} = -\frac{1}{4} \quad ; \quad \frac{\partial^2 N_3}{\partial \xi \partial \eta} = \frac{1}{4} \quad ; \quad \frac{\partial^2 N_4}{\partial \xi \partial \eta} = -\frac{1}{4}$$

di conseguenza:

$$\frac{\partial^2 x}{\partial \xi \partial \eta} = \sum_{i=1}^{4} \frac{\partial^2 N_i}{\partial \xi \partial \eta} x_i = \frac{x_1 - x_2 + x_3 - x_4}{4} \quad ; \quad \frac{\partial^2 y}{\partial \xi \partial \eta} = \sum_{i=1}^{4} \frac{\partial^2 N_i}{\partial \xi \partial \eta} y_i = \frac{y_1 - y_2 + y_3 - y_4}{4}$$

Ne consegue che:

$$\begin{cases} \dfrac{\partial f(\xi,\eta)}{\partial \xi} = \dfrac{\partial g(x,y)}{\partial \xi}\dfrac{\partial x}{\partial \xi} + \dfrac{\partial r(x,y)}{\partial \xi}\dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial h(\xi,\eta)}{\partial \eta} = \dfrac{\partial g(x,y)}{\partial \eta}\dfrac{\partial x}{\partial \eta} + \dfrac{\partial r(x,y)}{\partial \eta}\dfrac{\partial y}{\partial \eta} \\ \dfrac{\partial f(\xi,\eta)}{\partial \eta} = \dfrac{\partial g(x,y)}{\partial \eta}\dfrac{\partial x}{\partial \xi} + \dfrac{\partial r(x,y)}{\partial \eta}\dfrac{\partial y}{\partial \xi} + g(x,y)\dfrac{\partial^2 x}{\partial \xi \partial \eta} + r(x,y)\dfrac{\partial^2 y}{\partial \xi \partial \eta} \end{cases}$$

Poiché le funzioni $g(x,y)$ ed $r(x,y)$ dipendono delle variabili x ed y, abbiamo:

$$\frac{\partial g(x,y)}{\partial \xi} = \frac{\partial g(x,y)}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial g(x,y)}{\partial y}\frac{\partial y}{\partial \xi} \quad ; \quad \frac{\partial g(x,y)}{\partial \eta} = \frac{\partial g(x,y)}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial g(x,y)}{\partial y}\frac{\partial y}{\partial \eta}$$

$$\frac{\partial r(x,y)}{\partial \xi} = \frac{\partial r(x,y)}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial r(x,y)}{\partial y}\frac{\partial y}{\partial \xi} \quad ; \quad \frac{\partial r(x,y)}{\partial \eta} = \frac{\partial r(x,y)}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial r(x,y)}{\partial y}\frac{\partial y}{\partial \eta}$$

Sostituendo:

$$\begin{cases} \dfrac{\partial f(\xi,\eta)}{\partial \xi} = \left[\dfrac{\partial g(x,y)}{\partial x}\dfrac{\partial x}{\partial \xi} + \dfrac{\partial g(x,y)}{\partial y}\dfrac{\partial y}{\partial \xi}\right]\dfrac{\partial x}{\partial \xi} + \left[\dfrac{\partial r(x,y)}{\partial x}\dfrac{\partial x}{\partial \xi} + \dfrac{\partial r(x,y)}{\partial y}\dfrac{\partial y}{\partial \xi}\right]\dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial h(\xi,\eta)}{\partial \eta} = \left[\dfrac{\partial g(x,y)}{\partial x}\dfrac{\partial x}{\partial \eta} + \dfrac{\partial g(x,y)}{\partial y}\dfrac{\partial y}{\partial \eta}\right]\dfrac{\partial x}{\partial \eta} + \left[\dfrac{\partial r(x,y)}{\partial x}\dfrac{\partial x}{\partial \eta} + \dfrac{\partial r(x,y)}{\partial y}\dfrac{\partial y}{\partial \eta}\right]\dfrac{\partial y}{\partial \eta} \\ \dfrac{\partial f(\xi,\eta)}{\partial \eta} = \left[\dfrac{\partial g(x,y)}{\partial x}\dfrac{\partial x}{\partial \eta} + \dfrac{\partial g(x,y)}{\partial y}\dfrac{\partial y}{\partial \eta}\right]\dfrac{\partial x}{\partial \xi} + \left[\dfrac{\partial r(x,y)}{\partial x}\dfrac{\partial x}{\partial \eta} + \dfrac{\partial r(x,y)}{\partial y}\dfrac{\partial y}{\partial \eta}\right]\dfrac{\partial y}{\partial \xi} + g(x,y)\dfrac{\partial^2 x}{\partial \xi \partial \eta} + r(x,y)\dfrac{\partial^2 y}{\partial \xi \partial \eta} \end{cases}$$

da cui:

$$\begin{cases} \dfrac{\partial f(\xi,\eta)}{\partial \xi} = \left[\dfrac{\partial g(x,y)}{\partial x}\left(\dfrac{\partial x}{\partial \xi}\right)^2 + \dfrac{\partial g(x,y)}{\partial y}\dfrac{\partial x}{\partial \xi}\dfrac{\partial y}{\partial \xi}\right] + \left[\dfrac{\partial r(x,y)}{\partial x}\dfrac{\partial x}{\partial \xi}\dfrac{\partial y}{\partial \xi} + \dfrac{\partial r(x,y)}{\partial y}\left(\dfrac{\partial y}{\partial \xi}\right)^2\right] \\ \dfrac{\partial h(\xi,\eta)}{\partial \eta} = \left[\dfrac{\partial g(x,y)}{\partial x}\left(\dfrac{\partial x}{\partial \eta}\right)^2 + \dfrac{\partial g(x,y)}{\partial y}\dfrac{\partial x}{\partial \eta}\dfrac{\partial y}{\partial \eta}\right] + \left[\dfrac{\partial r(x,y)}{\partial x}\dfrac{\partial x}{\partial \eta}\dfrac{\partial y}{\partial \eta} + \dfrac{\partial r(x,y)}{\partial y}\left(\dfrac{\partial y}{\partial \eta}\right)^2\right] \\ \dfrac{\partial f(\xi,\eta)}{\partial \eta} = \left[\dfrac{\partial g(x,y)}{\partial x}\dfrac{\partial x}{\partial \xi}\dfrac{\partial x}{\partial \eta} + \dfrac{\partial g(x,y)}{\partial y}\dfrac{\partial x}{\partial \xi}\dfrac{\partial y}{\partial \eta}\right] + \left[\dfrac{\partial r(x,y)}{\partial x}\dfrac{\partial x}{\partial \eta}\dfrac{\partial y}{\partial \xi} + \dfrac{\partial r(x,y)}{\partial y}\dfrac{\partial y}{\partial \xi}\dfrac{\partial y}{\partial \eta}\right] + g(x,y)\dfrac{\partial^2 x}{\partial \xi \partial \eta} + r(x,y)\dfrac{\partial^2 y}{\partial \xi \partial \eta} \end{cases}$$

Risulta quindi:

$$\begin{cases} \dfrac{\partial f(\xi,\eta)}{\partial \xi} = \dfrac{\partial g(x,y)}{\partial x}\left(\dfrac{\partial x}{\partial \xi}\right)^2 + \left[\dfrac{\partial r(x,y)}{\partial x} + \dfrac{\partial g(x,y)}{\partial y}\right]\dfrac{\partial x}{\partial \xi}\dfrac{\partial y}{\partial \xi} + \dfrac{\partial r(x,y)}{\partial y}\left(\dfrac{\partial y}{\partial \xi}\right)^2 \\ \dfrac{\partial h(\xi,\eta)}{\partial \eta} = \dfrac{\partial g(x,y)}{\partial x}\left(\dfrac{\partial x}{\partial \eta}\right)^2 + \left[\dfrac{\partial r(x,y)}{\partial x} + \dfrac{\partial g(x,y)}{\partial y}\right]\dfrac{\partial x}{\partial \eta}\dfrac{\partial y}{\partial \eta} + \dfrac{\partial r(x,y)}{\partial y}\left(\dfrac{\partial y}{\partial \eta}\right)^2 \\ \dfrac{\partial f(\xi,\eta)}{\partial \eta} = \dfrac{\partial g(x,y)}{\partial x}\dfrac{\partial x}{\partial \xi}\dfrac{\partial x}{\partial \eta} + \dfrac{\partial r(x,y)}{\partial x}\dfrac{\partial x}{\partial \eta}\dfrac{\partial y}{\partial \xi} + \dfrac{\partial g(x,y)}{\partial y}\dfrac{\partial x}{\partial \xi}\dfrac{\partial y}{\partial \eta} + \dfrac{\partial r(x,y)}{\partial y}\dfrac{\partial y}{\partial \xi}\dfrac{\partial y}{\partial \eta} + g(x,y)\dfrac{\partial^2 x}{\partial \xi \partial \eta} + r(x,y)\dfrac{\partial^2 y}{\partial \xi \partial \eta} \end{cases}$$

Sostituendo le espressioni delle funzioni f, g, h e r e scambiando l'ordine della 2° e della 3° riga abbiamo:

$$\begin{cases} \dfrac{\partial^2 N_i}{\partial \xi^2} = \dfrac{\partial^2 N_i}{\partial x^2}\left(\dfrac{\partial x}{\partial \xi}\right)^2 + 2\dfrac{\partial^2 N_i}{\partial x \partial y}\dfrac{\partial x}{\partial \xi}\dfrac{\partial y}{\partial \xi} + \dfrac{\partial^2 N_i}{\partial y^2}\left(\dfrac{\partial y}{\partial \xi}\right)^2 \\ \dfrac{\partial^2 N_i}{\partial \xi \partial \eta} = \dfrac{\partial^2 N_i}{\partial x^2}\dfrac{\partial x}{\partial \xi}\dfrac{\partial x}{\partial \eta} + \dfrac{\partial^2 N_i}{\partial x \partial y}\left(\dfrac{\partial x}{\partial \eta}\dfrac{\partial y}{\partial \xi} + \dfrac{\partial x}{\partial \xi}\dfrac{\partial y}{\partial \eta}\right) + \dfrac{\partial^2 N_i}{\partial y^2}\dfrac{\partial y}{\partial \xi}\dfrac{\partial y}{\partial \eta} + \dfrac{\partial N_i}{\partial x}\dfrac{\partial^2 x}{\partial \xi \partial \eta} + \dfrac{\partial N_i}{\partial y}\dfrac{\partial^2 y}{\partial \xi \partial \eta} \\ \dfrac{\partial^2 N_i}{\partial \eta^2} = \dfrac{\partial^2 N_i}{\partial x^2}\left(\dfrac{\partial x}{\partial \eta}\right)^2 + 2\dfrac{\partial^2 N_i}{\partial x \partial y}\dfrac{\partial x}{\partial \eta}\dfrac{\partial y}{\partial \eta} + \dfrac{\partial^2 N_i}{\partial y^2}\left(\dfrac{\partial y}{\partial \eta}\right)^2 \end{cases}$$

Ricordando che:

$$\begin{Bmatrix} \dfrac{\partial N_i}{\partial x} \\ \dfrac{\partial N_i}{\partial y} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix}^{-1} \begin{Bmatrix} \dfrac{\partial N_i}{\partial \xi} \\ \dfrac{\partial N_i}{\partial \eta} \end{Bmatrix} = \frac{1}{\dfrac{\partial x}{\partial \xi}\dfrac{\partial y}{\partial \eta} - \dfrac{\partial x}{\partial \eta}\dfrac{\partial y}{\partial \xi}} \begin{bmatrix} \dfrac{\partial y}{\partial \eta} & -\dfrac{\partial y}{\partial \xi} \\ -\dfrac{\partial x}{\partial \eta} & \dfrac{\partial x}{\partial \xi} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial N_i}{\partial \xi} \\ \dfrac{\partial N_i}{\partial \eta} \end{Bmatrix} = \frac{1}{\|J\|} \begin{Bmatrix} \dfrac{\partial y}{\partial \eta}\dfrac{\partial N_i}{\partial \xi} - \dfrac{\partial y}{\partial \xi}\dfrac{\partial N_i}{\partial \eta} \\ -\dfrac{\partial x}{\partial \eta}\dfrac{\partial N_i}{\partial \xi} + \dfrac{\partial x}{\partial \xi}\dfrac{\partial N_i}{\partial \eta} \end{Bmatrix}$$

abbiamo:

$$\frac{\partial N_i}{\partial x}\frac{\partial^2 x}{\partial \xi \partial \eta} + \frac{\partial N_i}{\partial y}\frac{\partial^2 y}{\partial \xi \partial \eta} = \frac{1}{\|J\|}\left[\left(\frac{\partial y}{\partial \eta}\frac{\partial N_i}{\partial \xi} - \frac{\partial y}{\partial \xi}\frac{\partial N_i}{\partial \eta}\right)\frac{\partial^2 x}{\partial \xi \partial \eta} + \left(-\frac{\partial x}{\partial \eta}\frac{\partial N_i}{\partial \xi} + \frac{\partial x}{\partial \xi}\frac{\partial N_i}{\partial \eta}\right)\frac{\partial^2 y}{\partial \xi \partial \eta}\right] = C_i(\xi,\eta)$$

Sostituendo:

$$\begin{cases} \dfrac{\partial^2 N_i}{\partial \xi^2} = \dfrac{\partial^2 N_i}{\partial x^2}\left(\dfrac{\partial x}{\partial \xi}\right)^2 + 2\dfrac{\partial^2 N_i}{\partial x \partial y}\dfrac{\partial x}{\partial \xi}\dfrac{\partial y}{\partial \xi} + \dfrac{\partial^2 N_i}{\partial y^2}\left(\dfrac{\partial y}{\partial \xi}\right)^2 \\ \dfrac{\partial^2 N_i}{\partial \xi \partial \eta} = \dfrac{\partial^2 N_i}{\partial x^2}\dfrac{\partial x}{\partial \xi}\dfrac{\partial x}{\partial \eta} + \dfrac{\partial^2 N_i}{\partial x \partial y}\left(\dfrac{\partial x}{\partial \eta}\dfrac{\partial y}{\partial \xi} + \dfrac{\partial x}{\partial \xi}\dfrac{\partial y}{\partial \eta}\right) + \dfrac{\partial^2 N_i}{\partial y^2}\dfrac{\partial y}{\partial \xi}\dfrac{\partial y}{\partial \eta} + C_i(\xi,\eta) \\ \dfrac{\partial^2 N_i}{\partial \eta^2} = \dfrac{\partial^2 N_i}{\partial x^2}\left(\dfrac{\partial x}{\partial \eta}\right)^2 + 2\dfrac{\partial^2 N_i}{\partial x \partial y}\dfrac{\partial x}{\partial \eta}\dfrac{\partial y}{\partial \eta} + \dfrac{\partial^2 N_i}{\partial y^2}\left(\dfrac{\partial y}{\partial \eta}\right)^2 \end{cases}$$

Questa espressione si può scrivere nella seguente forma matriciale:

$$\begin{Bmatrix} \dfrac{\partial^2 N_i}{\partial \xi^2} \\ \dfrac{\partial^2 N_i}{\partial \xi \partial \eta} - C_i(\xi,\eta) \\ \dfrac{\partial^2 N_i}{\partial \eta^2} \end{Bmatrix} = \begin{bmatrix} \left(\dfrac{\partial x}{\partial \xi}\right)^2 & 2\dfrac{\partial x}{\partial \xi}\dfrac{\partial y}{\partial \xi} & \left(\dfrac{\partial y}{\partial \xi}\right)^2 \\ \dfrac{\partial x}{\partial \xi}\dfrac{\partial x}{\partial \eta} & \dfrac{\partial x}{\partial \eta}\dfrac{\partial y}{\partial \xi} + \dfrac{\partial x}{\partial \xi}\dfrac{\partial y}{\partial \eta} & \dfrac{\partial y}{\partial \xi}\dfrac{\partial y}{\partial \eta} \\ \left(\dfrac{\partial x}{\partial \eta}\right)^2 & 2\dfrac{\partial x}{\partial \eta}\dfrac{\partial y}{\partial \eta} & \left(\dfrac{\partial y}{\partial \eta}\right)^2 \end{bmatrix} \begin{Bmatrix} \dfrac{\partial^2 N_i}{\partial x^2} \\ \dfrac{\partial^2 N_i}{\partial x \partial y} \\ \dfrac{\partial^2 N_i}{\partial y^2} \end{Bmatrix} = [H] \begin{Bmatrix} \dfrac{\partial^2 N_i}{\partial x^2} \\ \dfrac{\partial^2 N_i}{\partial x \partial y} \\ \dfrac{\partial^2 N_i}{\partial y^2} \end{Bmatrix}$$

da cui:

$$\begin{Bmatrix} \dfrac{\partial^2 N_i}{\partial x^2} \\ \dfrac{\partial^2 N_i}{\partial x \partial y} \\ \dfrac{\partial^2 N_i}{\partial y^2} \end{Bmatrix} = [H]^{-1} \begin{Bmatrix} \dfrac{\partial^2 N_i}{\partial \xi^2} \\ \dfrac{\partial^2 N_i}{\partial \xi \partial \eta} - C_i(\xi,\eta) \\ \dfrac{\partial^2 N_i}{\partial \eta^2} \end{Bmatrix}$$

Indicando con $\begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}$ lo jacobiano, allora i coefficienti della matrice $[H]$ valgono:

$$H_{11} = \left(\frac{\partial x}{\partial \xi}\right)^2 = J_{11}^2 \quad ; \quad H_{12} = 2\frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \xi} = 2J_{11}J_{12} \quad ; \quad H_{13} = \left(\frac{\partial y}{\partial \xi}\right)^2 = J_{22}^2$$

$$H_{21} = \frac{\partial x}{\partial \xi}\frac{\partial x}{\partial \eta} = J_{11}J_{21} \quad ; \quad H_{22} = \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi} + \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} = J_{21}J_{12} + J_{11}J_{22} \quad ; \quad H_{23} = \frac{\partial y}{\partial \xi}\frac{\partial y}{\partial \eta} = J_{12}^2$$

$$H_{31} = \left(\frac{\partial x}{\partial \eta}\right)^2 = J_{21}^2 \quad ; \quad H_{32} = 2\frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \eta} = 2J_{21}J_{22} \quad ; \quad H_{33} = \left(\frac{\partial y}{\partial \eta}\right)^2 = J_{22}^2$$

In conclusione i coefficienti della matrice $[\boldsymbol{B}]$ assumono la forma seguente:

$$[\boldsymbol{B}] = \begin{bmatrix} \cdots & \frac{\partial^2 N_i}{\partial x^2} & \cdots \\ \cdots & \frac{\partial^2 N_i}{\partial y^2} & \cdots \\ \cdots & \frac{\partial^2 N_i}{\partial x \partial y} & \cdots \end{bmatrix} \quad \text{dove:} \quad \begin{Bmatrix} \frac{\partial^2 N_i}{\partial x^2} \\ \frac{\partial^2 N_i}{\partial x \partial y} \\ \frac{\partial^2 N_i}{\partial y^2} \end{Bmatrix} = [H]^{-1} \left\{ \begin{matrix} \frac{\partial^2 N_i}{\partial \xi^2} \\ \frac{\partial^2 N_i}{\partial \xi \partial \eta} \\ \frac{\partial^2 N_i}{\partial \eta^2} \end{matrix} - C_i(\xi, \eta) \right\}$$

Quando l'elemento ha la forma di un parallelogramma (ma solo in questo caso):

$$\frac{\partial^2 x}{\partial \xi \partial \eta} = \sum_{i=1}^{4} \frac{\partial^2 N_i}{\partial \xi \partial \eta} x_i = \frac{x_1 - x_2 + x_3 - x_4}{4} = 0 \quad ; \quad \frac{\partial^2 y}{\partial \xi \partial \eta} = \sum_{i=1}^{4} \frac{\partial^2 N_i}{\partial \xi \partial \eta} y_i = \frac{y_1 - y_2 + y_3 - y_4}{4} = 0$$

e di conseguenza: $C_i(\xi, \eta) = 0$. Le derivate seconde delle funzioni di forma rispetto alle variabili naturali ξ e η valgono:

$$\frac{\partial^2 N_1}{\partial \xi^2} = \frac{3\xi(1-\eta)}{4} \qquad \frac{\partial^2 N_1}{\partial \xi \partial \eta} = \frac{(4 - 3\xi^2 - 3\eta^2)}{8} \qquad \frac{\partial^2 N_1}{\partial \eta^2} = \frac{(1-\xi)3\eta}{4}$$

$$\frac{\partial^2 N_2}{\partial \xi^2} = \frac{(-1+3\xi)(1-\eta)}{4} \qquad \frac{\partial^2 N_2}{\partial \xi \partial \eta} = \frac{(1 + 2\xi - 3\xi^2)}{8} \qquad \frac{\partial^2 N_2}{\partial \eta^2} = 0$$

$$\frac{\partial^2 N_3}{\partial \xi^2} = 0 \qquad \frac{\partial^2 N_3}{\partial \xi \partial \eta} = \frac{(1 + 2\eta - 3\eta^2)}{8} \qquad \frac{\partial^2 N_3}{\partial \eta^2} = \frac{(1-\xi)(-1+3\eta)}{4}$$

$$\frac{\partial^2 N_4}{\partial \xi^2} = \frac{3\xi(-1+\eta)}{4} \qquad \frac{\partial^2 N_4}{\partial \xi \partial \eta} = \frac{(-4 + 3\xi^2 + 3\eta^2)}{8} \qquad \frac{\partial^2 N_4}{\partial \eta^2} = \frac{(1+\xi)3\eta}{4}$$

$$\frac{\partial^2 N_5}{\partial \xi^2} = \frac{(1+3\xi)(1-\eta)}{4} \qquad \frac{\partial^2 N_5}{\partial \xi \partial \eta} = \frac{(1 - 2\xi - 3\xi^2)}{8} \qquad \frac{\partial^2 N_5}{\partial \eta^2} = 0$$

$$\frac{\partial^2 N_6}{\partial \xi^2} = 0 \qquad \frac{\partial^2 N_6}{\partial \xi \partial \eta} = \frac{(-1 - 2\eta + 3\eta^2)}{8} \qquad \frac{\partial^2 N_6}{\partial \eta^2} = \frac{(1+\xi)(-1+3\eta)}{4}$$

$$\frac{\partial^2 N_7}{\partial \xi^2} = \frac{3\xi(-1-\eta)}{4} \qquad \frac{\partial^2 N_7}{\partial \xi \partial \eta} = \frac{(4 - 3\xi^2 - 3\eta^2)}{8} \qquad \frac{\partial^2 N_7}{\partial \eta^2} = \frac{(-1-\xi)3\eta}{4}$$

$$\frac{\partial^2 N_8}{\partial \xi^2} = \frac{(1+3\xi)(1+\eta)}{4} \qquad \frac{\partial^2 N_8}{\partial \xi \partial \eta} = \frac{(-1 + 2\xi + 3\xi^2)}{8} \qquad \frac{\partial^2 N_8}{\partial \eta^2} = 0$$

$$\frac{\partial^2 N_9}{\partial \xi^2} = 0 \qquad \frac{\partial^2 N_9}{\partial \xi \partial \eta} = \frac{(-1 + 2\eta + 3\eta^2)}{8} \qquad \frac{\partial^2 N_9}{\partial \eta^2} = \frac{(1+\xi)(1+3\eta)}{4}$$

$$\frac{\partial^2 N_{10}}{\partial \xi^2} = \frac{3\xi(1+\eta)}{4} \qquad \frac{\partial^2 N_{10}}{\partial \xi \partial \eta} = \frac{(-4 + 3\xi^2 + 3\eta^2)}{8} \qquad \frac{\partial^2 N_{10}}{\partial \eta^2} = \frac{(-1+\xi)3\eta}{4}$$

$$\frac{\partial^2 N_{11}}{\partial \xi^2} = \frac{(-1+3\xi)(1+\eta)}{4} \qquad \frac{\partial^2 N_{11}}{\partial \xi \partial \eta} = \frac{(-1 - 2\xi + 3\xi^2)}{8} \qquad \frac{\partial^2 N_{11}}{\partial \eta^2} = 0$$

$$\frac{\partial^2 N_{12}}{\partial \xi^2} = 0 \qquad \frac{\partial^2 N_{12}}{\partial \xi \partial \eta} = \frac{(1 - 2\eta - 3\eta^2)}{8} \qquad \frac{\partial^2 N_{12}}{\partial \eta^2} = \frac{(1-\xi)(1+3\eta)}{4}$$