Summer School on
ODEs with Discontinuous Right-Hand Side: Theory and Applications
Dobbiaco (BZ) – Italy

Sliding Mode Control:
Basic Theory, Advances and Applications

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Lecture 2

Higher Order Sliding Mode Control in Variable Structure Systems

- Higher Order Sliding Modes in Variable Structure Systems
- Control problem statement
- Dynamic Higher Order Sliding Mode Control
- Terminal Higher Order Sliding Mode Control
- Single-Input 2\textsuperscript{nd} Order Sliding Mode Control
- Single-Input 3\textsuperscript{rd} Order Sliding Mode Control
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L2 – HOSM in VSS

Classical sliding mode are characterized by constraining the state evolution onto a surface of the state space by means of a switching control

$$\dot{x} = f(x, u, t) \quad x \in \mathcal{G} \subset \mathbb{R}^n \quad u \in \mathbb{R}^q$$

$$\mathcal{G} = \{x \in \mathbb{R}^n : \sigma(x, t) = 0\} \quad \sigma \in \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^q$$

When the differential inclusion defining the closed loop dynamics belongs to the tangential space of the surface $\mathcal{G}$, a 2\textsuperscript{nd} Order Sliding Mode (2-Sliding Mode) appears

$$\dot{x} = f(x, u, t) \quad x \in \mathcal{G}_2 \subset \mathbb{R}^n \quad u \in \mathbb{R}^q$$

$$\mathcal{G}_2 = \{x \in \mathbb{R}^n : \sigma(x, t) = \dot{\sigma}(x, t) = 0\} \quad \sigma \in \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^q$$

The 2\textsuperscript{nd} Order Sliding Mode is characterized by a $n-2q$ reduced order dynamics
The constraint $\dot{\sigma}(x,t) = 0$ depends on the system dynamics as

$$\dot{\sigma}(x,t) = \frac{\partial \sigma}{\partial x} \cdot f(x,u,t) + \frac{\partial \sigma}{\partial t}$$
L2 – HOSM in VSS

Example

\[ \begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= f(x, u, t) \\
y &= x_1 + x_2 
\end{align*} \]

\[
\begin{bmatrix}
y \\
\dot{y} \\
w
\end{bmatrix} =
\begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
\]

\[
\begin{align*}
\dot{y} &= x_3 + f(y, \dot{y}, w, u, t) \\
\dot{w} &= -w + y
\end{align*}
\]
The definition of Higher Order Sliding Mode can be extended to a r-order sliding surface

\[ \dot{x} = f(x, u, t) \quad x \in \mathcal{G}_r \subset \mathbb{R}^n, \quad u \in \mathbb{R}^q \]

\[ \mathcal{G}_r = \left\{ x \in \mathbb{R}^n : \frac{d^k \sigma(x, t)}{dt^k} = 0, k = 0, 1, \ldots, r - 1 \right\} \quad \sigma \in \mathbb{R}^n \times \mathbb{R}^+ \rightarrow \mathbb{R}^q \]

\( \sigma \) is a sufficiently smooth vector function: \( \sigma \in \mathcal{C}^{r-1} \)

Discontinuity appears in the \( r \)-th order time derivative of \( \sigma \)

The overall number of constraints must be less than the system order

\[ n > rq \]
L2 – HOSM in VSS

Definition

Let the $r$-sliding set $\mathcal{G}_r$ be non-empty and assume that it is locally an integral set in Filippov’s sense (i.e. it consists of Filippov’s trajectories of the discontinuous dynamic system). Then the corresponding motion of the system state belonging to the set $\mathcal{G}_r$ is called $r$-Sliding Mode (r-SM) with respect to the constraint function $\sigma$. 
A Higher–Order Sliding Mode Control (HOSMC) system is implemented when the control \( u \) is able to constrain the system state onto the set \( \mathcal{G}_r \) starting from any point in a \( \varepsilon \)-vicinity of the set \( \mathcal{G}_r \).

Finding a control law such that it is able to enforce a \( r \)-SM on a surface in the state space is a difficult task for generic nonlinear systems

\[
\dot{x} = f(x, u, t)
\]

A simpler system dynamics is usually considered
L2 – HOSMC: Problem statement

\[ \dot{x} = \bar{f}(\bar{x}, \bar{u}, t) \]

\[
\begin{bmatrix}
\dot{\bar{x}} \\
\dot{\bar{u}} \\
\dot{x}_0 \\
\dot{x}
\end{bmatrix} =
\begin{bmatrix}
\dot{\bar{x}} \\
1 \\
0 \\
0
\end{bmatrix} \begin{bmatrix}
\hat{x} \\
1 \\
0 \\
0
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
I_q \\
0
\end{bmatrix}u
\]

\[ \dot{x} = f(x) + g(x)u \]

**f(x):** drift vector function  
**x:** state variables vector  
**g(x):** gain matrix function  
**u:** control variables vector
L2 – HOSMC: Problem statement

Notation

\[ h(x) : \mathbb{R}^n \rightarrow \mathbb{R} \quad \nabla h(x) = \frac{\partial h}{\partial x} \quad \text{Gradient vector} \]

\[ h(x) : \mathbb{R}^n \times \mathbb{R}^q \rightarrow \mathbb{R}^p \quad \nabla h(x) = \begin{bmatrix} \nabla h_1(x) \\ \vdots \\ \nabla h_p(x) \end{bmatrix} = J^h \quad \text{Jacobian matrix} \]

\[ h(x) : \mathbb{R}^n \rightarrow \mathbb{R} \quad L_g^k h = \nabla \left( L_g^{k-1} h \right) \cdot g(x) \quad i = 1,2,\ldots \quad L_g^0 h = h(x) \quad \text{Lie derivatives} \]

\[ h(x) : \mathbb{R}^n \rightarrow \mathbb{R}^p \quad L_g^k h = \nabla \left( L_g^{k-1} h \right) \cdot g(x) = J^h \cdot g(x) \]

\[ g(x) : \mathbb{R}^n \rightarrow \mathbb{R}^n \quad L_g^k h = \nabla \left( L_g^{k-1} h \right) \cdot g(x) \quad i = 1,2,\ldots \quad L_g^0 h = h(x) \]
The Higher Order Sliding Mode Control problem is to define a suitable output function $\sigma(t)$ and a proper control law $u(t)$ such that the state trajectory of the dynamical system

$$\dot{x} = f(x) + g(x)u$$

is constrained onto the r-order sliding surface

$$\mathcal{G}_r = \left\{ x \in \mathbb{R}^n : \frac{d^k \sigma(x)}{dt^k} = 0, k = 0, 1, \ldots, r-1 \right\}, \quad \sigma \in \mathbb{R}^n \times \mathbb{R}^r \rightarrow \mathbb{R}^q, \quad u \in \mathbb{R}^q$$

from a time instant $t_\infty$ on, possibly in finite time ($t_\infty \leq T < \infty$).
L2 – HOSMC: Problem statement

Given the control affine system

\[ \dot{x} = f(x) + g(x)u \]

\[ x \in \mathbb{R}^n \quad u \in \mathbb{R}^q \]

\[ f : \mathbb{R}^n \to \mathbb{R}^n \quad g : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^q \]

A \( r \)-order Sliding Mode Control on the surface \( \sigma(x) = 0 \) can be designed if the following conditions are fulfilled

\[ L_g L_f^i \sigma = 0 \quad i = 0, 2, \ldots, r - 2 \]

The control does not appear in the first \( r-1 \) derivatives of the system output \( \sigma(x) \)

\[ L_g L_f^{r-1} \sigma \quad \text{has full rank} \]

The output variable \( \sigma(x) \) is completely controllable by the \( u(x) \) control
L2 – HOSMC: Problem statement

Several HOSMC algorithms were presented in the literature

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L2 – Dynamic HOSMC

Dynamic Higher Order sliding Mode Control is based on the definition of an augmented output variable

\[ s(x) = \sigma^{(r\text{-}1)} + \sum_{i=0}^{r-2} c_i \sigma^{(i)}(x) \]

Coefficients \(c_i\) are chosen so that the polynomial \(P(p)\) has all roots with negative real part

\[ P(p) = p^{r-1} + \sum_{i=0}^{r-2} c_i p^i \]

The control \(u(t)\) affects the time derivative of \(s(x)\), and if it is designed so that \(s(x)\) is stabilized to zero, each output variable \(s_i\), and its \(r\text{-}1\) time derivatives, are stabilized to zero asymptotically.
Theorem.
Consider the system dynamics
\[ \dot{x} = f(x) + g(x)u \]

Chose the sliding variable set \( \sigma(x) \) such that the internal dynamics corresponding to the output variables \( \sigma, \dot{\sigma}, \ddot{\sigma}, \ldots, \sigma^{(r-1)} \) is BIBS stable.

Assume that the uncertainties are bounded by a known function, and that the gain matrix of the input-output dynamics is definite positive

\[ \left\| L^k_t \sigma \right\| \leq \Lambda_k(x) \quad \text{for } k = 1,2,\ldots,r \]
\[ L^k g L^k f \sigma = 0 \quad \text{for } k = 0,1,\ldots,r-2 \]
\[ 0 < \Lambda_m \leq \min\{eig[L^r g L^{r-1} f \sigma]\} \quad \forall \ x \in \Omega, \forall \ t \]

Define an auxiliary output as a stable linear combination of the sliding variable and its \( r-1 \) time derivatives

\[ s(x) = \sigma^{(r-1)} + \sum_{i=0}^{r-2} c_i \sigma^{(i)}(x) \]

The state feedback control law assures the asymptotic stability of the \( r \)-SM on the sliding surface \( \sigma(x) = 0 \)

\[ u = - \frac{\Lambda_r(x) + \sum_{i=0}^{r-2} c_i \Lambda_{i+1}(x) + \eta}{\Lambda_m \left\| s \right\|_2} \quad \text{for } \eta > 0 \]
**L2 – Dynamic HOSMC**

**Proof.**

Auxiliary sliding variable dynamics

\[
\dot{s}(t) = L_t^r \sigma + \sum_{i=0}^{r-2} c_i L_t^{i+1} \sigma + L_g L_t^{r-1} \sigma \cdot u(t)
\]

Lyapunov function

\[
V(s) = \frac{1}{2} s^T \cdot s
\]

\[
\dot{V} = \sigma^T \cdot \dot{\sigma}
\]

\[
= s^T \cdot \left[ L_t^r \sigma + \sum_{i=0}^{r-2} c_i L_t^{i+1} \sigma - \frac{\Lambda_r(x) + \sum_{i=0}^{r-2} c_i \Lambda_{i+1}(x) + \eta}{\Lambda_m} L_g L_t^{r-1} \sigma \cdot \frac{s}{\|s\|_2} \right]
\]

\[
\leq -\frac{\eta}{\Lambda_m \|s\|_2} s^T \cdot L_g L_t^{r-1} \sigma \cdot s \leq -\eta \|s\|_2 < -\eta \sqrt{V} < 0
\]

Once the condition \( s=0 \) is fulfilled, each sliding variable motion is characterized by a **asymptotically** stable linear dynamics.
L2 – Dynamic HOSMC

Dynamic Higher Order Sliding Modes are an extension of the classic 1-SM

The Higher Order Sliding Mode behavior is reached asymptotically

Perfect knowledge of the time derivatives of the sliding variable up to the r-1 order are needed

\[ \dot{\sigma} + c_0\sigma = 0 \]
Terminal 2\textsuperscript{nd}-Order sliding Mode Control is based on the properties of finite
time control by means of the definition of an augmented nonlinear output variable

\[
s(x) = \sigma(x) + \frac{1}{\beta} \dot{\sigma}^{p/m}
\]

Coefficients \(p \) and \(m \) \((p>m)\) are odd numbers so that the motion on the
hyper-surface \(s = 0\) in the phase space is characterized by the finite time
reaching of the origin

The control \(u(t)\) affects the time derivative of \(s(x)\), and if it is designed so that
\(s(x)\) is stabilized to zero, each output variable \(s_i\), and its \(r-1\) time derivatives,
are stabilized to zero asymptotically
L2 – Terminal 2-SMC

**Theorem.**
Consider the system dynamics
\[ \dot{x} = f(x) + g(x)u \]

Chose the sliding variable set \( \sigma(x) \) such that the internal dynamics corresponding to the output variables \( \sigma, \dot{\sigma} \) is BIBS stable.

Assume that the uncertainties are bounded by a known function and the gain matrix of the input-output dynamics is definite positive

\[
\begin{aligned}
\| L_f^2 \sigma \| &\leq \Lambda_2(x) \\
L_g \sigma &= 0 \\
\exists \left[ L_g L_f^{-1} \sigma \right]^{-1}
\end{aligned}
\]

Define an auxiliary output as a **stable** nonlinear combination of the sliding variable and its time derivatives

\[ s(x) = \sigma(x) + \frac{1}{\beta} \dot{\sigma}^{p/m} \]

The state feedback control law assures the finite time stability of the 2-SM on the sliding surface \( \sigma(x) = 0 \)

\[
u = -\left[ L_g L_f^{-1} \sigma \right]^{-1} \left[ \beta \frac{m}{p} \dot{\sigma}^{2-p/m} + (\Lambda_2 + \eta) \text{sgn}(s) \right] \quad \eta > 0
\]
Proof.

Auxiliary sliding variable dynamics

\[
\dot{s}(t) = \dot{\sigma} + \frac{1}{\beta} \frac{p}{m} \text{diag}\{\dot{\sigma}_i^{2-\nu/m}\} \left( L_f^2 \sigma + L_g L_f \sigma \cdot u(t) \right)
\]

Lyapunov function

\[
V(s) = \frac{1}{2} s^T \cdot s
\]

\[
\dot{V} = \frac{1}{\beta} \frac{p}{m} s^T \cdot \text{diag}\{\dot{\sigma}_i^{2-\nu/m}\} \left( L_f^2 \sigma + L_g L_f \sigma \cdot u(t) \right)
= \frac{1}{\beta} \frac{p}{m} s^T \cdot \text{diag}\{\dot{\sigma}_i^{2-\nu/m}\} \left[ L_f^2 \sigma - \left( \beta \frac{m}{p} \dot{\sigma}^{2-\nu/m} + (\Lambda_2 + \eta) \text{sgn}(s) \right) \right]
\leq -\frac{\eta}{\beta} \frac{p}{m} |s|^T \cdot \dot{\sigma}^{\nu/m-1} < 0 \quad \forall \dot{\sigma} \neq 0
\]

\[
\dot{\sigma} \leq -\eta \text{sgn}(\sigma) \quad \forall \dot{\sigma} = 0
\]

Once the condition \(s=0\) is fulfilled, each sliding variable motion is characterized by a **finite time** stable linear dynamics.
Terminal 2\textsuperscript{nd} Order Sliding Modes are the multi-variable non-singular implementation of the “prescribed law of variation” 2-SMC

The 2\textsuperscript{nd} Order Sliding Mode behavior is reached in finite time

Perfect knowledge of the first time derivatives of the sliding variable order is needed
L2 – Single Input 2-SMC

Both dynamic HOSM and terminal 2-SM are implementations of the classical sliding mode control approach but with a peculiar sliding manifold.

Several algorithms that implement a 2\textsuperscript{nd} Order Sliding Mode directly have been presented in the literature.

They are based on peculiar trajectories on the $\sigma - \dot{\sigma}$ plane.

Because of the lack of Lyapunov like methods for proving their finite time stability, they are usually presented with reference to Single Input systems.

$$\dot{x} = f(x) + g(x)u \quad x \in \mathcal{G} \subset \mathbb{R}^n \quad u \in \mathbb{R}$$

$$\mathcal{G} = \{ x \in \mathbb{R}^n : \sigma(x,t) = 0 \} \quad \sigma \in \mathbb{R}^n \times \mathbb{R}^+ \rightarrow \mathbb{R}$$
There are two main families of algorithms:

Supertwisting: it is designed for relative degree 1 sliding surface (the control $u$ appears in the first time derivative of the sliding variable)

The parameter tuning is based on the knowledge of an upper bound for the time derivative of the drift term

Only the sliding variable has to be available

Sub-Optimal: it is designed for relative degree 2 sliding surface (the control $u$ appears in the second time derivative of the sliding variable)

The parameter tuning is based on the knowledge of an upper bound for the time derivative of the drift term and it can be dynamically set to have global convergence properties

The sliding variable has to be available, and possibly also the sign of its time derivative
\[
\dot{\sigma} = \varphi(\bullet) + \gamma(\bullet)u
\]
\[
u = -\alpha(t)U\sgn(\sigma - \beta \sigma_M) \\
\alpha(t) = \begin{cases} 
1 & (\sigma - \beta \sigma_M)\sigma_M \geq 0 \\
\alpha^{-} & (\sigma - \beta \sigma_M)\sigma_M < 0 
\end{cases}
\]
\[
\beta \in (0,1) \\
\sigma \in [1;+\infty) \cap \left(\frac{2\Phi + \Gamma_M U(1-\beta)}{\Gamma_m U(1+\beta)};+\infty\right)
\]

\(\Phi\): upper bound of the drift term  
\(\Gamma_m, \Gamma_M\): positive lower and upper bounds of the gain term

The control is discontinuous

The control is based on the estimation of the local maxima, minima and first order flex points
L2 – Single Input 2-SMC – Sub Optimal

\[ u = -\alpha(t)U \text{sgn}(\sigma - \beta \sigma_M) \]
\[ \alpha(t) = \begin{cases} 1 & (\sigma - \beta \sigma_M)\sigma_M \geq 0 \\ \bar{\alpha} & (\sigma - \beta \sigma_M)\sigma_M < 0 \end{cases} \]
\[ \beta \in (0;1) \]

\( \alpha \): modulation parameter
\( \beta \): anticipation parameter
\( U \): gain parameter

The proper tuning of the anticipation parameter allows for counteracting the peaking phenomenon.
\[ u = -\alpha(t)U \text{sgn}(\sigma - \beta \sigma_M) \]

\[ \alpha(t) = \begin{cases} 
1 & (\sigma - \beta \sigma_M)\sigma_M \geq 0 \\
\frac{1}{\alpha} & (\sigma - \beta \sigma_M)\sigma_M < 0 
\end{cases} \]

\[ \beta \in (0;1) \]

— increasing \( V_M \)

— anticipation \( \beta \)

— modulation \( 1/\alpha \)

reference \( \Box \)
**Theorem.**
Consider the system dynamics \( \dot{x} = f(x) + g(x)u \)

Chose the sliding variable set \( \sigma(x) \) such that the internal dynamics corresponding to the output variables \( \sigma, \dot{\sigma} \) is BIBS stable.

Assume that that the sliding dynamics has relative degree 2 and constant bounds for the uncertain drift and gain vector are known

Set the controller parameters such that the convergence conditions are fulfilled

The state feedback control law assures the asymptotic stability of the 2-SM on the sliding surface \( \sigma(x) = 0 \)
**Proof.**

\[ U > \frac{\phi}{\Gamma_m} \]

**Dominance condition**

\[ \bar{\alpha} \in [1;+\infty) \cap \left( \frac{2\phi + \Gamma_M U(1 - \beta)}{\Gamma_m U(1 + \beta)} ; +\infty \right) \]

**Convergence condition**

\[ |\sigma_{M_{k+1}}| \leq \rho |\sigma_{M_k}|, \quad \rho \in (0,1) \]

\[ |\dot{\sigma}_{M_k}| \leq \sqrt{2(1 - \beta)(\Gamma_M U + \Phi)} |\sigma_{M_k}|, \]

A geometric convergent series of extremal points is generated and the 2-SM is achieved in a finite time.
The Generalized Sub-Optimal 2\textsuperscript{nd} Order Sliding Mode Control algorithm is based on a switching logic based on the memory of past values of the sliding variable.

The values $\sigma_{M,k}$ can be ideally evaluated by looking at the past, or by checking the time instant at which the time derivative of the sliding variable vanishes.

The reaching phase can be characterized by twisting around the origin of the $\sigma, \dot{\sigma}$ plane, or by monotonic convergence of the sliding variable.

The Generalized Sub-Optimal algorithm can implement the Sub-Optimal ($\beta = \frac{1}{2}$) and the Twisting algorithms by proper tuning of the parameters.

**Condition for monotonic convergence**

$$\bar{\alpha} \in \left[1, +\infty \right) \cap \left( \frac{\Phi + (1 - \beta) \Gamma_M U}{\beta \Gamma_m U}; +\infty \right)$$

**Condition for Twisting algorithm**

$$\bar{\alpha} > \frac{\beta = 0}{2 \Phi + \Gamma_M U \Gamma_m U}$$
L2 – Single Input 2-SMC – Sub Optimal

The parameters of the Generalized Sub-Optimal 2\textsuperscript{nd} Order Sliding Mode Control algorithm can be adjusted to guarantee the global convergence property.

Global convergence can be lost because of the peaking phenomenon that is usually related to large values of the time derivative of the controlled output.

Anticipating the switching will cause a reduction of the values

\[
\left| \dot{\sigma}_{M_k} \right| \leq \sqrt{2(1 - \beta)(\Gamma_M U + \Phi)} \left| \sigma_{M_k} \right|
\]

Very small values of $\beta$ will cause very slow convergence to the 2-SM

\[
\tau_\infty \leq \tau_0 + U \frac{\bar{\alpha}\Gamma_m + \Gamma_M}{\bar{\alpha}\Gamma_m U - \Phi} \sqrt{\frac{2(1 - \beta)}{\Phi + \Gamma_M U}} \left[ \frac{1}{\Phi + [(1 - \beta)\Gamma_M - \bar{\alpha}\beta\Gamma_m]U} \right] - \frac{1}{\bar{\alpha}\Gamma_M U - \Phi}
\]
The gain $U$ and the anticipation $\beta$ of the Generalized Sub-Optimal 2\textsuperscript{nd} Order Sliding Mode Control algorithm are adjusted at any local extremal value of $\sigma$

$$|L^2_t\sigma| \leq \phi(\sigma, \sigma^*)$$
$$0 < \Gamma_m \leq L_g L_t \sigma \leq \Gamma_M(\sigma, \sigma^*)$$

$$\beta_k = \max\left\{\frac{1}{2}, 1 - \frac{\eta^2}{2(\bar{\phi}_k + \bar{\Gamma}_M U_M)}\right\}$$

$$u = -U_{M_k} \text{sign}\left(\sigma - \beta_k \sigma_M\right)$$

$$U_{M_k} > \frac{1}{\bar{\Gamma}_M}\left(\bar{\phi}_k + \frac{1}{3}\eta^2\right)$$

$$\bar{\phi}_k = \phi(\sigma_M, \eta \sqrt{\sigma_M})$$
$$\bar{\Gamma}_M = \Gamma_M(\sigma_M, \eta \sqrt{\sigma_M})$$

A prediction of the steepest transient is performed to evaluate the right control gain and anticipation in order to limit the magnitude of the time derivative of $\sigma$
L2 – Single Input 2-SMC – Sub Optimal

An illustrative example

The red line shows the case of a more restrictive condition on the time derivative of the sliding variable
$\dot{\sigma} = \varphi(\dot{\sigma}) + \gamma(\dot{\sigma})u$

$$ u = -\lambda \sqrt{|\sigma|} \text{sgn}(\sigma) - \alpha \int \text{sgn}(\sigma) dt $$

$$ \lambda^2 > 2 \frac{\Phi + \alpha \Gamma_M}{\Gamma_m} $$

$\Phi$: upper bound of the “drift” term

$\Gamma_m, \Gamma_M$: positive lower and upper bounds of the gain term

The control is continuous

The control is based on the measure of the sliding variable only
L2 – Single Input 2-SMC – Supertwisting

**Theorem.**
Consider the system dynamics  
\[
\dot{x} = f(x) + g(x)u
\]

Chose the sliding variable set \( \sigma(x) \) such that the internal dynamics corresponding to the output variables \( \sigma, \dot{\sigma} \) is BIBS stable.

Assume that the sliding dynamics has relative degree 1 and constant bounds for the uncertain drift and gain vector are known

Set the controller parameters such that the convergence conditions are fulfilled

The state feedback control law assures the asymptotic stability of the 2-SM on the sliding surface \( \sigma(x) = 0 \)
Proof.

Differential inclusion

\[ U > \frac{\Phi}{\Gamma_m} \]

Dominance condition

Convergence condition

\[
\begin{aligned}
&\{L_f^2\sigma + (L_f L_g + L_g L_f)u + u^2 L_g^2 \sigma \} \leq \Phi \\
&0 < \Gamma_m \leq L_g \sigma \leq \Gamma_M \\
&\alpha > \frac{\Phi}{\Gamma_m}, \quad \lambda^2 > 2 \frac{\Phi + \alpha \Gamma_M}{\Gamma_m}
\end{aligned}
\]

\[
\left| \dot{\sigma}_{M_{k+1}} \right| \leq \rho \left| \dot{\sigma}_{M_k} \right|, \quad \rho \in (0,1)
\]

A geometric convergent series of extremal points is generated and the 2-SM is achieved in a finite time
L2 – Single Input 2-SMC – Supertwisting

The Supertwisting 2\textsuperscript{nd} Order Sliding Mode Control algorithm is based on mixing a nonlinear finite time first order feedback and an asymptotic second order switching logic.

The reaching phase are characterized by a smooth twisting around the origin of the $\sigma, \dot{\sigma}$ plane.

The Supertwisting algorithm is effective for SISO systems having relative degree 1.

The 2_SM feature allows for easy implementation of an exact finite time convergent first order differentiator.
The 3rd Order Sliding Mode Control refers to systems whose sliding variable has relative degree 3 with respect to the (discontinuous) control variable

\[ L_g L_i^k \sigma = 0 \quad i = 0,1 \quad \text{for} \quad L_g L_i^2 \sigma \neq 0 \]

The main point is to design a control law with the less measurement demand as possible

Find a finite-time stabilizer that depends only on the sign of the phase variables

\[ u(t) = u[\text{sign}(\sigma), \text{sign}(\dot{\sigma}), \text{sign}(\ddot{\sigma})] \]
L2 – Single Input 3-SMC

The considered 3\textsuperscript{rd} order dynamics has known constant bounds

\[ |L^3_t \sigma| \leq \phi \]

\[ L_g L^k_t \sigma = 0 \quad i = 0,1 \]

\[ 0 < \Gamma_m \leq L_g L^2_t \sigma \leq \Gamma_M \]

A “switched” sliding-mode controller commuting between two structures:

**Anosov unstable [AU]**

\[ u = -U \text{sign}(\sigma) \]

\[ U > \phi / \Gamma_m \]

**Generalized Twisting [G-TW]**

\[ u = -U_1 \text{sign}(\dot{\sigma}) - U_2 \text{sign}(\ddot{\sigma}) \]

\[ \Lambda_m [U_1 + U_2] - \phi > \Gamma_M [U_1 - U_2] + \phi \]
L2 – Single Input 3-SMC

What is the system behaviour under the separate action of the two considered “structures”?

Anosov unstable

$u = -U_0 \text{sign}(\sigma)$

The appearance of a sequence of zero crossings is ensured.

$\exists t_{z_i} \quad i = 1, 2, \ldots, \infty$

$: \quad \sigma(t_{z_i}) = 0$

The sliding variable features unstable oscillations around zero.

$\sigma(t_{z_i}) \rightarrow \infty$
L2 – Single Input 3-SMC

What is the system behaviour under the separate action of the two considered “structures”?

Generalized Twisting
G-TW

\[ u = -\frac{1}{2}(U_1 + U_2)\text{sign}(\dot{s}) - \frac{1}{2}(U_1 - U_2)\text{sign}(\ddot{s}) \]

\( \dot{\sigma} \) and \( \ddot{\sigma} \) both converge to zero in finite time

\( \sigma \) converges to a constant value \( \sigma_e \) (not necessarily zero)

\((\sigma_e,0,0)\) is said to be an “equilibrium point”
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L2 – Single Input 3-SMC

The 3\textsuperscript{rd} Order Sliding Mode Controller switches between the two structure according the represented automaton.

- AU always leads to a zero crossing
- G-TW always leads to an eq. point
- The “event-driven” controller is never “blocked”
**Theorem.**
Consider the system dynamics 
\[ \dot{x} = f(x) + g(x)u \]

Chose the sliding variable set \( \sigma(x) \) such that the internal dynamics corresponding to the output variables \( \sigma, \dot{\sigma}, \dot{\sigma} \) is BIBS stable.

Assume that that the sliding dynamics has relative degree 3 and constant bounds for the uncertain drift and gain vector are known

Define the parameters \( \Delta_i \)

\[ \begin{align*}
\left\| L_f^3 \sigma \right\| &\leq \Phi \\
L_g \sigma = L_g L_f \sigma &= 0 \quad \forall \ x \in \mathcal{V}, \forall \ t \\
0 < \Gamma_m &\leq L_g L_f^2 \sigma \leq \Gamma_M
\end{align*} \]

\[ \begin{align*}
\Delta_1 &= \frac{\Gamma_M U_0 + \Phi}{\Gamma_m U_1 - \Phi} \\
\Delta_2 &= \frac{\Gamma_M U_0 + \Phi}{\Gamma_m U_2 - \Phi} \\
\Delta_3 &= \frac{\Gamma_m U_1 - \Phi}{\Gamma_M U_2 + \Phi} \\
\Delta_4 &= \frac{\Gamma_m U_2 + \Phi}{\Gamma_m U_2 - \Phi}
\end{align*} \]

Choosed the contraction rate 
\[ \varepsilon \in (0,1) \]
Theorem, cont.

The state feedback control law assures the asymptotic stability of the 3-SM on the sliding surface $\sigma(x) = 0$, if the controller parameters are chosen according the following design steps

1) Set

2) Set $U_2$ sufficiently large so that

3) Set $U_1$ sufficiently large so that

$$u = -\frac{1}{2}(U_1 + U_2)\text{sign}(\dot{s}) - \frac{1}{2}(U_1 - U_2)\text{sign}(\ddot{s})$$

$$U_0 > \frac{\Phi}{\Gamma_m}$$

$$0 < 2\sqrt{\Delta_2} < \varepsilon$$

$$\Delta_1(3 + 2\Delta_1) + 2\sqrt{\Delta_2}(1 + 2\Delta_1)^{3/2} < \varepsilon$$

$$\frac{\Lambda_1^2 - 1}{\Lambda_3} > \Lambda_4^2$$
L2 – Single Input 3-SMC

The proof is based on finding the conditions on the control parameters such that a series of contracting equilibrium points is enforced.

\[ |\sigma_{e,i+1}| \leq \varepsilon |\sigma_{e,i}| \quad i = 0,1,2,\ldots \]

A typical time-evolution of the sliding quantity
The Hybrid 3rd Order Sliding Mode Control algorithm is based on switching between an unstable controller and a stable controller that in general force the closed loop system onto an equilibrium point with $\sigma \neq 0$

A reduced amount of information is required

A specific care must be devoted to identify the reaching of an equilibrium point

Tuning is quite easy if the step of the procedure are carefully performed
L2 – Arbitrary Order SMC

The Arbitrary Order Sliding Mode Control laws are based on a recursive algorithm deriving from the “prescribed law of variation” 2-SMC

A r-SMC algorithm needs the knowledge of the sliding variable, of its r-2 time derivatives and the sign of its r-1 time derivative

It is almost always associated to an arbitrary order differentiator

As for the Dynamical Sliding Mode Control it is assumed that the time derivatives of the sliding variable are available

Several implementation were presented trying to get a smoother reaching phase
Consider the nonlinear system

\[ \dot{x} = f(x) + g(x)u \]

Define a suitable output \( \sigma(x) \)

\[ L_g L_i^k \sigma = 0 \quad i = 1, 2, \ldots, r - 2 \]

\[ L_g L_i^{r-1} \sigma \neq 0 \]

Calculate \( m \) as the least common multiplier of \( 1, 2, 3, \ldots, r \)

Define the quantities

\[ N_{i,r} = \left( \sum_{k=0}^{i-1} \sigma^{(k)} \right)^{r-1/m} \quad i = 1, 2, \ldots, r - 1 \]

Define the hyper-surfaces

\[ \Phi_{0,r} = \sigma \]

\[ \Phi_{i,r} = \sigma^{(i)} + \beta_i N_{i,r} \text{sgn}(\Phi_{i-1,r}) \quad i = 1, 2, \ldots, r - 1 \]
L2 – Arbitrary Order SMC

From the definitions it is clear that

\[ \Phi_{i,r} = 0 \Rightarrow \Phi_{i-1,r} \rightarrow 0 \quad i = 1, 2, \ldots, r - 1 \]

Example \( r = 3 \)

\[ m = 2 \cdot 3 = 6 \]

\[ N_{1,r} = \left| \sigma \right|^{\frac{1}{3}} \]

\[ N_{2,r} = \left( \left| \sigma \right|^{\frac{1}{3}} + \left| \dot{\sigma} \right|^{\frac{1}{2}} \right)^{\frac{1}{6}} \]

\[ \Phi_{0,r} = \sigma \]

\[ \Phi_{1,r} = \dot{\sigma} + \beta_1 \left| \sigma \right|^{\frac{1}{3}} \text{sgn}(\sigma) \]

\[ \Phi_{2,r} = \dot{\sigma} + \beta_2 \left( \left| \sigma \right|^{\frac{1}{3}} + \left| \dot{\sigma} \right|^{\frac{1}{2}} \right)^{\frac{1}{6}} \text{sgn}(\dot{\sigma} + \beta_1 \left| \sigma \right|^{\frac{1}{3}} \text{sgn}(\sigma)) \]
L2 – Arbitrary Order SMC

If the sliding dynamics is bounded

\[
\begin{align*}
\|L_t^r \sigma\| & \leq \Phi \\
0 < \Gamma_m & \leq L_g L_t^{r-1} \sigma \leq \Gamma_M
\end{align*}
\]

The differential inclusion defining the sliding variable dynamics is

\[
\sigma^{(r)} \in [-\Phi, +\Phi] + [\Gamma_m, \Gamma_M] u
\]

The controller

\[
u = -\alpha \ \text{sgn}\left(\Phi_{r-1,r}\left(\sigma, \dot{\sigma}, \ldots, \sigma^{(r-1)}\right)\right)
\]

with parameters \(\alpha\) and \(\beta_i\) sufficiently large will force \(\Phi_{r-1,r}\) to zero in finite time.

Then the “cascade” convergence will be established.
L2 – Arbitrary Order SMC

The coefficients appearing on each $\Phi_{i,r}$ are such that the closed loop dynamics has homogeneity properties

A vector–set field $F(x) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is called homogeneous of degree $q \in \mathbb{R}$ with the dilation $d_\kappa : (x_1, x_2, \ldots, x_n) \rightarrow (x_1^{m_1}, x_2^{m_2}, \ldots, x_n^{m_n})$, $m_i \in \mathbb{R}^+$, if the following identity holds

$$F(x) = \kappa^{-q} d_\kappa^{-1} F(d_\kappa x) \quad \forall \kappa > 0$$

A differential inclusion $\dot{x} \in F(x)$ is called homogeneous of degree $q \in \mathbb{R}$ if it is invariant with respect to the combined time–coordinate transformation $\Delta_\kappa : (t, x) \rightarrow (\kappa^{-q}, d_\kappa x)$
The homogeneity property implies the finite time stability once the contraction is proved.

Coefficients $\beta_i$ can be defined once and the only tuning parameter is $\alpha$ depending on the bounds of the uncertain dynamics.

\[
\begin{align*}
  r &= 1. \quad u = -\alpha \text{sign}(s) \\
  r &= 2. \quad u = -\alpha \text{sign}(\dot{s} + |s|^{1/2} \text{sign } s) \\
  r &= 3. \quad u = -\alpha \text{sign}(\dot{s} + 2(|\dot{s}|^3 + |s|^2)^{1/6} \text{sign}(\dot{s} + |s|^{2/3} \text{sign } s)) \\
  r &= 4. \quad u = -\alpha \text{sign}\left\{s^{(3)} + 3(\dot{s}^6 + \dot{s}^4 + |s|^3)^{1/12} \text{sign}[\dot{s} + \\
  &\quad (\dot{s}^4 + |s|^3)^{1/6} \text{sign}(\dot{s} + 0.5|s|^{3/4} \text{sign } s)]\right\}
\end{align*}
\]
L2 – References


