This chapter introduces Resampled Efficient Frontier (REF) optimization, a generalization of linear constrained Markowitz MV portfolio optimization that includes uncertainty in investment information in the optimization process. Monte Carlo resampling methods are used to more realistically condition investment information in the optimization. REF optimality avoids the literal use of investment information characteristic of classical MV and other portfolio optimization methods. Under practical assumptions, REF optimization is provably effective at improving linear constrained risk-adjusted portfolio return on average. REF optimality typically leads to a more effective level of diversification and risk management than previously available. The resampling process also allows customization of the optimization process relative to investment mandates, objectives, strategies, and information character.

**EFFICIENT FRONTIER STATISTICAL ANALYSIS**

The enormous variation of Monte Carlo efficient frontier simulations in Exhibit 5.1 demonstrates the need for a statistical view of MV efficiency.

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1. RE optimization was invented by Richard Michaud and Robert Michaud and is a U.S. patented procedure, worldwide patents pending. It was originally described in Michaud (1998, Chapter 6). New Frontier Advisors, LLC (NFA) is exclusive worldwide licensee.

2. This chapter addresses sign-constrained portfolio optimization, generally the framework of choice for institutional asset allocation. Index-relative and long-short optimization, often used in institutional equity portfolio management, is treated in Chapter 9.
In Exhibit 5.1, every simulated MV efficient frontier is the right way to invest for a given set of inputs. However, the inputs are highly uncertain: how should an investor use the portfolio optimality uncertainty displayed in the exhibit? From one perspective, the instability of MV efficiency with estimation error demonstrated in the exhibit may indicate little hope of practical investment value. In reality, the variation suggests a statistical route for transforming MV optimization into a more investment useful procedure.

For a highly risk-averse investor, the minimum variance portfolio is the optimal portfolio for any simulated efficient frontier. Since all simulated efficient frontiers are equally likely, Resampled Efficiency™ (RE) defines the optimal minimum variance portfolio as the average of the portfolio weights of all the simulated minimum variance portfolios. Exhibit 6.1 shows the optimal minimum variance portfolio plotted at the base of the lower, REF, curve in Exhibit 6.1.

For a risk-indifferent investor, the maximum return portfolio is the optimal portfolio for any given simulated efficient frontier. Since all simulated maximum return portfolios are equally likely, the RE optimal maximum return portfolio is defined as the average of the portfolio weights of all the simulated maximum return portfolios and is plotted at the top of the lower curve in Exhibit 6.1.

Similarly, RE optimality can be defined for the utility function that characterizes any investor’s risk-return preferences. The average of the maximum expected utility tangent portfolios on simulated MV efficient
frontiers in Exhibit 5.1 defines the RE optimal portfolio. The REF plots as the lower curve in Exhibit 6.1 and is the collection of all possible RE optimal portfolios with risk aversion parameters from expected utility curves ranging from total risk aversion to total risk indifference. The exhibit shows that the REF lies within the range of estimation error alternative optimal portfolios as shown in Exhibit 5.2. The results demonstrate that REF portfolios reflect safer, less extreme investments when uncertainty in investment information is considered. More generally, the REF is based on averages of all properly associated optimal portfolios on the simulated MV efficient frontiers.

Table 6.1 displays the minimum variance, middle, and maximum average return portfolios from the two efficient frontiers in Exhibit 6.1. Little difference exists between RE and MV optimal portfolios at very low risk. Of course, this is a simple example consisting of only eight assets; larger optimization universes may find important differences. As indicated in columns 4 and 5 of Table 6.1, moderate-risk portfolios display more pronounced optimality differences; the RE portfolio is more diversified and has less extreme large and small allocations. Columns 6 and 7 in Table 6.1 display the dramatic difference between the maximum return MV and RE portfolios; the MV portfolio is a single asset while the RE portfolio is an investment-intuitive, diversified portfolio.

3. This construction process is consistent with λ-association as described in the appendix. The process highlights how rational agents may make investment decisions that lead to REF optimality. A concern (e.g., Markowitz & Usmen, 2003) that the REF is not consistent with rational agent decision making may be due to the original description of the procedure in the Michaud (1998) text that used rank-association, a heuristic construction process, seemingly devoid of utility considerations. Rank-order association is used as a convenient compute-efficient approximation to utility-based REF construction. Our views on rationality axioms and rule-based systems are discussed further in Michaud (2003, fn. 6).

4. The examples in the text use rank-order association for computing the REF portfolios. The illustrations are based on computing 51 portfolios equally spaced from low to high return for the classical and each simulated efficient frontier. The RE portfolio is computed as the average of the rank-associated simulated MV efficient portfolios. The REF portfolios are the collection of the RE portfolios associated with an MV efficient frontier. The procedure is a useful, simple, compute-efficient, and statistically stable estimate of utility-function-based REF portfolios. Other approximations are also available with various compute-efficiency and statistical stability characteristics.

5. Mathematically, REF optimality is an integral in portfolio space of the expected value of the MV optimal portfolio weights. The resampling/bootstrap process is a Monte Carlo method for spanning for a given level of uncertainty the linear-constraint-defined portfolio space probabilistically and estimating the integral.

6. It may be of interest to note a statistical perspective on the innovation implicit in the definition of the REF. Resampling and bootstrap methods in statistics are generally concerned with exploring the variability implicit in historical data, as in Efron (2005). RE optimization uses the variability exposed by resampling to define a new statistic that did not exist before.

7. In this case, there is a simple interpretation and analytical derivation of the RE maximum return optimal portfolio. Each asset weight is equal to the probability that it is truly the maximum return asset.
As indicated by Exhibit 6.1, REF portfolios lie below and generally well within the range of portfolio risk spanned by the MV efficient frontier. The REF is not a statistical artifact of portfolio simulation but represents a computable alternative set of investments. The question of interest is whether REF portfolios provide an investment-relevant and practical alternative for defining portfolio optimality.

From a superficial point of view, RE optimization appears to be an inferior investment framework. This is because the REF expects less return and has a more restricted range of risk relative to classical efficiency. In-sample studies of portfolio efficiency, such as Harvey et al. (2003), conclude that the REF does not define optimal portfolios. The apparent inferiority of REF portfolios provides the first glimpse, one of many, of the limitations of in-sample MV efficiency portfolio analysis.

The appropriate interpretation of REF versus classical MV optimality is straightforward. If you are 100% certain of your risk-return estimates (to 16 decimal places of accuracy or more), the Markowitz efficient frontier is the appropriate definition of portfolio efficiency. If you are less than 100% certain of your risk-return estimates, you expect less return and are less willing to put money at risk, and REF optimality is appropriate. The REF properly reflects portfolio optimization in the context of

8. In some cases the REF may extend beyond the MV efficient frontier. These cases are not material to our discussion here.
9. In-sample utility is greater for classical than REF portfolios. As discussed below, the Harvey et al. (2003) investor is unlikely to be pleased with their “more optimal” solution.
10. One of the most important contributions of REF analysis is the notion that in-sample MV efficiency analysis is an unreliable and often misleading framework for portfolio analysis. Implications for asset management are discussed specifically in Michaud and Michaud (2005b) and later in the text.
11. As noted in Chapter 4, computed MV efficient frontier portfolios reflect 16 decimal places of accuracy for most modern computers.
information uncertainty. To drive the point home, consider an investor
with a complete lack of certainty in his or her investment information.
In this case the optimal efficient frontier is the no-information prior
portfolio, either equal or benchmark weighted.\textsuperscript{12} The REF portfolio is the
no-information portfolio in this case, while Markowitz optimization
remains insensitive to information uncertainty.\textsuperscript{13} RE optimization is the
paradigm of choice for rational decision making under conditions of
information uncertainty.

As Exhibit 6.1 illustrates, the RE and MV frontiers may be close in MV
space. A superficial reading may suggest that the procedures produce
similar solutions. Exhibit 6.2 is a portfolio composition map of the MV
and RE optimal asset allocations in Exhibit 6.1. Each band of shading rep-
resents one of the eight assets in the base case. A vertical strip through
the bands provides the optimal portfolio allocations at that risk level.
The portfolios with minimum variance appear on the left-hand side of
the charts, high-return portfolios on the right. The upper panel presents
the composition map for MV efficiency; the lower panel depicts RE
optimality. In the left-hand side of each panel, the dark area represents
nearly a 100% allocation to Euro bonds at the low-risk end of the efficient
frontiers.

Exhibit 6.2 shows that the MV efficient frontier includes only five out
of the eight assets: Euro bonds and U.S., U.K., Japanese, and French equi-
ties. If the return estimate for U.K. equities (middle asset in the MV map)
is reduced by 0.5\%, the allocation to U.K. equities disappears across the
entire classical frontier. Since the standard error of the expected return
for the U.K. equities is much larger than 0.5\%, this result reflects far more
sensitivity than is desirable or sensible for a statistically insignificant
change. Similar, or more extreme, examples of statistical insensitivity
and instability can be found on nearly every MV frontier.

The composition map for the REF illustrates very different proper-
ties. REF optimality includes all eight assets. The allocations transition
smoothly from one risk level to another. A reduced estimate of return
for U.K. equities by 0.5\% produces a hardly noticeable change in opti-
amal allocations across the entire frontier. RE optimization is robust and
fundamentally different in character and allocations, even when the two
frontiers are similar in in-sample MV space.\textsuperscript{14}

\textsuperscript{12} It is a necessary condition that the risk spectrum for estimation error-sensitive MV portfolio effi-
ciency converges to the no-information portfolio as uncertainty increases. This property contradicts
the properties of the heuristic Feldman (2003) and Ceria and Stubbs (2005) methods, where the risk
spectrum is constant and equal to classical efficiency whatever the level of certainty in investment
information. It also contradicts the conclusions of the Chopra and Ziemba (1993) study. Our results
demonstrate that estimation error in risk as well as return is necessary for appropriately defining port-
folio optimality under information uncertainty.

\textsuperscript{13} Changing the level of forecast certainty in the RE optimization process is discussed further below.

\textsuperscript{14} Ad hoc portfolio constraints are often put in place to improve the stability of the MV optimized
solution. Ironically, they can often introduce instability rather than reduce it. Consider the following
TRUE AND ESTIMATED OPTIMIZATION INPUTS

Markowitz gives you the right way to invest given that you happen to know that your risk-return estimates are correct. Under these conditions, no other set of input assumptions or set of portfolios is more appropriate as a basis for investment. Although the Markowitz MV efficient frontier portfolios are not necessarily the investment performance winners for a given draw of returns in the investment period, on average they are the example: in an optimization of many assets, two assets (assets A and B) have similar risk and return characteristics. Suppose the optimizer weights assets A and B in similar proportion along the unconstrained frontier. If a binding upper bound were introduced for asset B, asset A would increase at a greater rate along the frontier to make up for the unavailable asset B. Similarly, if asset B is constrained but the forecast returns vary for asset A, the volatility of the optimal portfolio weight of asset A would be increased. Improperly implemented, constraints can form a knife-edge, forcing the optimizer to make sharp decisions and leading to greater portfolio weight instability. The RE optimization creates robust solutions by averaging all the knife-edge MV optimizations relative to the uncertainty in the information.

Exhibit 6.2 MV and RE Frontier Portfolio Composition Maps
best-performing for a given risk level for perfect certainty. In this case, REF optimality has little investment interest.

The problem with the scenario in the previous paragraph, and its conclusion, is that it is completely unrealistic. The true value of optimization inputs is unknown and unknowable. Risk-return estimates in practice include substantial estimation error and are at best an informed guess of the true values in the investment period. As demonstrated in Exhibits 5.1 and 5.2, what characterizes MV portfolio optimization is its extreme sensitivity to estimation errors. RE optimization addresses the issue of estimation error sensitivity intrinsic to MV efficiency.

One of the most attractive features of REF portfolios in practice is that they are often consistent with investment intuition without the need for ad hoc constraints. For example, the maximum return MV efficient portfolio in Exhibit 6.1 represents a 100% bet on French equities. However, the optimization inputs for Japanese and French equities in Table 2.3 are virtually identical. Based purely on these inputs, investors are likely to prefer an equal bet on both markets. In addition, from a return/risk basis, the inputs for U.K. and U.S. equities are not very different from Japanese and French equities. Consequently, the diverse RE efficient portfolio is preferable to the MV efficient 100% bet on French equities. The reduced range of risk simply reflects the need for more diverse optimal portfolios and is a direct consequence of the uncertainty in investment information ignored in classical efficiency.

RE portfolios depend less on any particular characteristic of the optimization inputs. They reflect less extreme portfolio weights than MV portfolios. Because REF portfolios are averages, not outliers, they are more likely to provide safe and reliable investments with better out-of-sample performance on average. Note that REF portfolios with more moderate bets on assets may have additional practical investment benefits, from reduced liquidity demands to lower trading costs in portfolio rebalancings.

SIMULATION PROOFS OF RESAMPLED EFFICIENCY OPTIMIZATION

RE optimized portfolios have many desirable investment properties. However, as we will show, one of the most important features of RE optimization is its provable performance superiority on average under

15. This point is just statistics. In any random draw of investment returns, there is a likelihood that actual events deviate from underlying population statistical parameters in the same way that tossing a fair coin 10 times will not always result in five heads.
16. Simulation studies implicitly assume that the return distribution reflected in the historic data is stationary for the investment horizon of interest. The non-stationarity of the return distribution adds another significant dimension of estimation error to portfolio optimization in practice.
17. Note that even in a two-asset optimization, REF optimality provides useful information by limiting risk taking at the high end of the frontier.
practical investment assumptions.\textsuperscript{18} Jobson and Korkie (1981) tested the investment performance of unbounded versus equal-weighted portfolios using simulation proofs. Their procedure can be used to test the investment performance of MV versus RE linearly constrained optimized portfolios.\textsuperscript{19}

In a simulation study, the referee is assumed to know the true set of risk-returns for the assets. For this simulation study, the base case data (Tables 2.3 and 2.4) represent the true risk-returns.\textsuperscript{20} The referee does not tell investors the true values but provides a set of Monte Carlo simulated returns consistent with the true risks and returns.\textsuperscript{21} In the base case data set, each simulation consists of 18 years of monthly returns and represents a possible out-of-sample realization of the true values of the optimization parameters. Each set of simulated returns results in an estimate (with estimation error) of the optimization parameters and an MV efficient frontier. Each MV efficient frontier and set of estimated optimization parameters defines an RE optimized frontier. This process is repeated many times.\textsuperscript{22} In each of the simulations of MV and RE optimized frontiers, the referee uses the true risk-return values to score the actual risks and returns of the optimized portfolios.

The averaged results of the simulation study are displayed in Exhibit 6.3. The upper dotted curves display the in-sample averaged MV and RE frontiers that were submitted to the referee for scoring. The higher dotted curve is the MV efficient frontier; the lower dotted curve is the REF. The portfolios are plotted based on the simulated risks and returns. However, the referee knows the true risks and returns for each simulated optimized portfolio. The bottom solid curves in Exhibit 6.3 display the average of the true, out-of-sample, risks and returns of the optimized portfolios. The higher solid curve represents the RE optimized results, the lower solid curve the Markowitz optimized results. The lower curves in the exhibit show that the RE optimized portfolios, on average, achieve roughly the

\textsuperscript{18} A simulation proof requires assumptions about the true distribution of assets. The “truth” data set has to be in good financial order for the simulation to properly represent a useful out-of-sample investment process. Historical data may not always reflect a financially relevant truth data set because it may often include dominated assets. For example, the monthly returns for the default data set of indices for the 10-year period from January 1996 to December 2005 exhibits a negative average return for the Japanese index. In this case Japan is a dominated asset relative to other assets in the optimization universe and investment in Japan in the context of sign constraints makes no sense. Dominated assets that are inconsistent with a relevant financial “truth” in the context of sign constraints for simulation study purposes need to be excluded. While RE optimization can’t be proven to have higher out-of-sample risk-return in all possible simulation tests, properly implemented it outperforms for financially relevant cases of practical interest. Appendix B provides a geometric proof of superiority that is data-set-independent. The tests in this chapter assume sign-constrained optimization. The tests in Chapter 9 treat the index-relative and associated long-short case.

\textsuperscript{19} Simulation tests are preferable relative to back tests since back tests are time period dependent and there are not enough observations available to test for statistical significance.

\textsuperscript{20} We are indebted to Olivier Ledoit for critical assistance in defining the test framework.

\textsuperscript{21} Our simulations assume multivariate normal returns.

\textsuperscript{22} A minimum of 500 simulations of the MV and RE frontiers is used in the study.
same return with less risk, or alternatively more return with the same level of risk, relative to the Markowitz portfolios. The results represent the average out-of-sample investment experience of an investor using either MV or RE portfolios.

The simulation experiment illustrates that the RE optimized portfolios are, on average, provably effective at improving risk-adjusted investment performance. RE optimized portfolios perform better because they are better risk-managed by avoiding the unrealistically literal use of investment information that characterizes Markowitz MV optimization.

It should be noted that the simulation results presented are very conservative. In practice estimation error is far more prevalent than that represented by a stationary distribution of simulated monthly returns over an 18-year investment period. Reducing the number of simulated

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23. Markowitz and Usmen (2003) replicated the results with the same data set.
24. Some notes need to accompany the proofs of enhanced investment value in Exhibit 6.3. The results assume total or real return, sign and budget constrained, MV optimized portfolios. The conclusions are generalizable for the leverage constraints typically imposed in practice. Leverage simply extends the frontiers. The index- or benchmark-relative case and associated long-short issues are discussed in Chapter 9. Note that because Euro bonds effectively dominate U.S. bonds on a risk-return basis, classical efficiency outperforms REF portfolios at extremely low risk. In this case, there is little ambiguity associated with the optimal minimum variance portfolio for this data set. In practice, however, Euro bond estimation error is unknown and an investor would be unable to rely on low-risk dominance in the investment period. This is an additional though more subtle example of the impact of including dominated assets in simulation studies.
25. Knight and Satchell (2006) find no benefit to RE optimization. However, they examine only the unbounded asset weight case as in Chapter 4.
returns increases estimation error to more realistic levels and enhances the relative benefits associated with RE optimized portfolios. More importantly, return distributions are not stationary in investment practice. The many scenarios simulated from investment information engineer a portfolio optimality designed to protect investments from unlikely perverse events.

WHY DOES IT WORK

In institutional asset allocation practice, optimization universes consist of investment-attractive assets. As Merton (1987) observes, the optimization universe should consist of what you know. Nonnegative sign constraints are consistent with an all-assets-investable prior. Frost and Savarino (1988) demonstrate that out-of-sample MV optimized portfolio performance may be enhanced by combining sign constraints with resampling data. An equally weighted portfolio is a candidate optimal allocation in this context.

Sign constraints impose valuable investment structure on each resampled MV efficient frontier. They act as Bayesian priors to create a bias in the structure of the resampled optimized portfolios, using the uncertainty of the resampling process to define candidate optimal portfolios. By definition, the average of the resampled MV efficient portfolios is not an outlier but reflects the uncertainty inherent in investment information. The simulations show that the averaging process leads to improved average out-of-sample performance.

CERTAINTY LEVEL AND RE OPTIMALITY

Up until now, each simulated MV efficient frontier discussed has been computed by simulating 18 years of monthly returns relative to the data in tables 2.3 and 2.4. The reason for simulating 18 years of monthly returns is to be consistent with the information level in the original data set. However, in general, investors do not know that their risk-return estimates reflect a specific level of information.

The number of simulated returns used to compute the simulated MV efficient frontiers is a free parameter of the RE optimization process. As the number of returns becomes large, the set of simulated risk-return

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26. Lottery tickets and postage stamps are asset classes typically ignored in an institutional asset allocation.
27. Including leverage requires a trivial change in the argument and does not invalidate our conclusions.
28. Moderation of extreme portfolio weights is a characteristic of Stein estimators, discussed in Chapter 8. Bayes estimation, discussed in Chapter 11, may also moderate extreme portfolio weights. In contrast to RE optimization, these alternatives operate by changing the inputs prior to initiating the optimization process.
29. Appendix B provides a geometric perception for understanding why RE optimization works.
Efficient Asset Management estimates approaches the original risk-return estimates and the REF approaches the MV efficient frontier. When the number of observations becomes small, the REF approaches the no-information prior efficient portfolio. The number of simulated returns is a natural parameter for modeling the confidence an investor has in risk-return estimates. Exhibit 6.4 illustrates RE optimal frontiers at different Forecast Confidence™ (FC) levels.\textsuperscript{30,31} As the level of certainty increases, the REF approaches classical MV efficiency. The notion of FC level leads to a fundamental insight: RE optimization is simply a generalization of Markowitz MV optimization that allows investors to control the amount of confidence they have in their investment information in the optimization process.

**FC LEVEL APPLICATIONS**

One of the most serious critiques of classical portfolio optimization is asset management rigidity. The market outlook is an important consideration for many investment managers. Different style managers use investment information very differently and often reflect very different views and valuations of similar assets. Yet classical optimization is indifferent to the character and source of investment information. Classical investment process rigidity is a key reason for the indifference or lack of confidence many institutional managers exhibit toward portfolio optimization technology.

In contrast, RE optimization is a flexible framework for portfolio optimization in asset management. In particular, the FC level is a valuable

\textsuperscript{30} Forecast Confidence level is a patent-pending procedure.\textsuperscript{31} To facilitate the user experience, the Forecast Confidence (FC) level scale ranges from 1 to 10, indicating very low to very high information level. On this scale Markowitz optimization is an 11 and complete uncertainty 0. See further discussion and applications in Michaud and Michaud (2004a).
tool for customizing RE optimization to a manager’s investment process. For example, variation in the FC level in the optimization may be used to reflect changes in confidence in the market outlook. Growth stock managers may wish to raise FC levels to reduce portfolio diversification, reflecting the more ephemeral near-term character of their information, while value managers may wish to lower FC levels to increase diversification, reflecting the longer-term character of their information. The ability to customize as well as create optimized investment strategies is a hallmark of the RE optimization process.

THE REF MAXIMUM RETURN POINT (MRP)

The statistical nature of RE portfolio optimization leads to some significant differences from classical MV optimization. For instance, the REF frontier may peak and then curve downward. This turning point is the “maximum return point” (MRP) of the REF. The possible existence of an MRP is a key concept for understanding and using REF optimality.

Exhibit 6.5 illustrates how the REF MRP may arise. In each panel there are three high-risk assets; uncertainty is indicated by the ellipse around each point. The left-hand panel presumes an MV optimization level of certainty in information; the risk-returns are point estimates. The efficient frontier MRP portfolio includes only one asset. As uncertainty increases, as in the middle panel, there is less certainty concerning the return of the highest-return/risk asset; the REF includes some allocation to all three assets and the MRP REF portfolio lies below and to the left of the Markowitz maximum return portfolio. In the right-hand panel there is little certainty in the return of the highest-risk assets and the REF includes significant allocations in all three assets. In this case the MRP portfolio may emerge where the REF has a downward-sloping inefficient

![image]

Exhibit 6.5 Forecast Certainty Levels and the REF Maximum Return Point

32. See Michaud and Michaud (2004b).
segment. Any risk beyond the MRP is not optimal and not on the REF by definition.

The REF MRP assures investors that the efficient maximum return portfolio is appropriately diversified. The MRP arises because RE optimization uses information from all assets in the optimization universe.\(^{33}\) In contrast, since there is no notion of risk-return estimation uncertainty in classical MV optimization, investors may think that taking increasing amounts of risk is always justifiable.\(^{34}\)

An REF MRP is relatively rare in institutional asset allocation studies. This is because assets in the optimization universe often have relatively similar attractive risk-return characteristics. In contrast, an REF MRP is often observed in large stock universe equity portfolio optimizations. This is because the optimization universe may have many assets that have relatively little return but much risk. The existence of high-risk assets with little return implies that high risk return is uncertain. As uncertainty increases, the uncertainty at high-risk portfolio levels reduces expected return and the REF exhibits an MRP. The identification of the REF MRP point prevents the overuse of inferior investments.\(^{35}\) A necessary condition for a well-defined equity optimizer is to estimate the maximum level of risk that is consistent with the level of information in the optimization universe.\(^{36}\)

There is an important associated issue in this context. Why would investors include statistically insignificant assets in the optimization universe? Merton (1987) teaches that the optimization universe should be defined in terms of investable assets. An optimizer is not capable of telling which investments are not investable; that is the role of the analyst. Including non-investable assets in an optimization is much like including bad information in a Bayesian prior. However, Merton’s advice in the context of an equity optimization for a large stock index may result in unacceptably large tracking error risk and underrepresented sectors and industries. One simple solution is described in Michaud and Michaud (2005a): include an index-weighted composite asset of the statistically insignificant stocks in the index in the optimization.\(^{37}\) The Merton principle of investing only in what you know remains the appropriate one.

\(^{33}\) Preliminary simulation tests are consistent with out-of-sample replication of in-sample REF MRPs.

\(^{34}\) The existence of the MRP is a useful way of justifying much institutional investment practice where assumed tracking error is often far less than the maximum available in a MV optimization. While active asset managers are sometimes critiqued for being closet indexers, a low level of active risk may only reflect a rational view of level of information in their estimates. Alternatively, not knowing the limits of efficient risk in an optimized portfolio may afflict asset management for many leveraged hedge fund managers.

\(^{35}\) Generally, low-return, high-risk securities have very small allocations in REF portfolios. More important is the financial rationale associated with their inclusion in the optimization.

\(^{36}\) The concept of the MRP has important applications in scaling returns and proper optimization design.

\(^{37}\) Further description and implications of using the composite asset are given in the reference and in Chapter 9.
IMPLICATIONS FOR ASSET MANAGEMENT

The REF plots below the classical frontier because it reflects uncertainty in investment information. As a consequence, the REF challenges much conventional academic and professional wisdom on optimality and management practice.

The REF challenges the results of many studies of in-sample utility function optimization of portfolios on the MV efficient frontier. Because REF portfolios have less estimated return and risk, in-sample utility studies find REF portfolios “less than optimal.” But investors are very unlikely to prefer the “more optimal” portfolios on the MV efficient frontier if they promise less likely risk-adjusted return ex post. The consequence of ignoring out-of-sample performance of optimized portfolios is that many conclusions of in-sample utility studies are likely to be misleading or invalid. Journal editors are well advised to require simulation studies of out-of-sample optimized portfolio performance in the context of estimation error as a matter of good practice.

The REF challenges the results of many studies based on analytical formulas derived from optimizing the in-sample information ratio (IR) or reward-to-risk ratio of portfolios on the unconstrained MV efficient frontier. In these studies the IR is used as an investment intuitive and practical surrogate for in-sample utility. In-sample studies for maximizing IR without estimation error lead to seductive though erroneous prescriptions for asset management, such as increasing the size of the optimization universe and trading frequently. While the analytic formulas for in-sample IR are improved, the out-of-sample consequences on performance of the optimized portfolios are ignored. Not only are the prescriptions likely to be invalid, they are often the inverse of good investment practice. Clearly, only a framework that ignores estimation error and out-of-sample performance could conclude that increasingly frequent trading would improve optimality. Our studies demonstrate that considering the implications of estimation error on out-of-sample performance is essential for defining portfolio optimality and avoiding serious investment practice errors.

CONCLUSION

RE is an important new tool for defining portfolio efficiency in practice. It is useful for understanding the statistical characteristics and

38. Examples include Harvey et al. (2003), who do not address out-of-sample performance of their “more optimal” utility functions, and Chopra and Ziemba (1993), who invalidly conclude that estimation error in risk can be ignored.
40. More discussion is given in Michaud and Michaud (2005b).
41. Trading issues are discussed further in Chapter 7.
42. The composite asset procedure of Chapter 9 and other considerations for dealing with statistically insignificant investment information often reverses the prescriptions of IR based analytic formula studies.
Efficient Asset Management

practical limitations of MV efficiency. In addition, in the context of a relevant constraint prior, it is provably effective on average at enhancing the out-of-sample investment value of optimized portfolios. Relative to MV efficiency, resampled efficient portfolios are also likely to be more robust and investment intuitive, two useful characteristics in many institutional contexts.

**APPENDIX A: RANK-VERSUS \(\lambda\)-ASSOCIATED RE PORTFOLIOS**

Rank association is used in the text for computing RE optimal portfolios. One simple alternative is to associate simulated MV efficient frontier portfolios using a quadratic utility function. Given a value of \(\lambda\) (lambda), associate the efficient and simulated efficient linear constrained portfolios that minimize:

\[
\phi = \sigma^2 - \lambda \cdot \mu. \tag{6.1}
\]

Each value of \(\lambda\) defines a specific portfolio on the MV and simulated efficient frontiers. Varying \(\lambda\) from zero to infinity spans the set of efficient and simulated efficient frontier portfolios. Table 6A.1 displays the true reward-to-risk ratios for MV and resampled efficiency in the same test procedure as in Exhibit 6.3, where \(\lambda\) is used to associate simulated with efficient portfolios. The \(\lambda\) values are shown in the first row of Tables 6A.1 and 6A.2.\(^{43}\) \(\lambda\)-association appears to be slightly less statistically stable than rank-association. While rank-association is not always the procedure of choice, it is often a practical compromise.

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<th>20</th>
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<td>0.202</td>
<td>0.200</td>
<td>0.196</td>
<td>0.173</td>
<td>0.161</td>
<td>0.151</td>
</tr>
<tr>
<td>Return (%)</td>
<td>3.4</td>
<td>4.5</td>
<td>5.1</td>
<td>5.8</td>
<td>8.1</td>
<td>9.0</td>
<td>9.6</td>
</tr>
<tr>
<td>Risk (%)</td>
<td>5.4</td>
<td>6.4</td>
<td>7.4</td>
<td>8.6</td>
<td>13.6</td>
<td>16.3</td>
<td>18.5</td>
</tr>
</tbody>
</table>

\(^{43}\) For example, when \(\lambda\) equals 10, 15, and 20 in Table 6A.2, the average true risk is larger for resampled than for MV efficiency.
APPENDIX B: ROBERT’S HEDGEHOG

The following example helps to illustrate RE optimization.

Robert has a favorite pet hedgehog named Ralph. Ralph escaped from his cage and is now somewhere in the tall grass surrounding the house. Fortunately, Robert fitted Ralph with a GPS locator device, and the house is surrounded by a sturdy hedgehog-proof fence that Ralph can’t burrow underneath. Therefore we can safely assume that Ralph is somewhere inside the fence, and we know that his GPS locator can pinpoint Ralph’s location within a 10-meter radius circle. To find the hedgehog as quickly as possible, though, Robert wants to start his search where Ralph is most likely to be. How does Robert find his hedgehog?

Referring to Exhibit 6.6, we see that the GPS locator shows Ralph is somewhere in the circle with center A. Point A is therefore a place to start Robert’s search. However Robert notes that A is outside of the fenced yard. So Robert narrows his search to the area indicated by the GPS system but within the fence. Points on the fence and within the circle close to point A are better starting points. Point M is the closest fenced point to A and is a more optimal place for Robert to begin his search for Ralph.\(^44\)

Suppose the information used by the GPS locator system is statistically estimated and has estimation error. In this case it is useful to

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\(^44\) Hedgehog picture courtesy of Florida Center for Instructional Technology.
repeatedly query the GPS system and request new resampled estimates of the hedgehog’s position that may vary materially with each estimate. Some of the resampled estimates similar to M may place Ralph at points along the fence; others would be in the yard’s interior. Since there is no reason to choose one of the estimates over the others, we take an average of all solutions on the fence and in the yard. The resampled estimate of Ralph’s location is likely to be near but not at the fence, as in point R. The resampled estimate is therefore likely to be a more realistic estimate of the actual position of Robert’s hedgehog.

Our story has important implications for estimation of optimized portfolios. Suppose Ralph represents not a hedgehog in a two-dimensional yard but an optimal portfolio in high-dimensional portfolio space, and the signal comes not from a GPS but from an estimate of the return distribution of all assets. Point A, outside the fence, represents an unconstrained MV optimization estimate. Point M, at the fence, represents a Markowitz constrained MV optimization solution. Resampling the constrained MV problem gives multiple solutions along the boundary and interior of the solution space. Point R, the average of these solutions, represents the RE optimization solution. R is a better, more realistic solution than M because it includes all the information associated with Markowitz optimization as well as addressing estimation error.

Of course, finding Ralph is a lot simpler than finding an optimal portfolio. We know things exactly in Ralph’s case that we can only estimate for a real problem. For example, we know exactly what the boundary and confidence region look like for Ralph but must estimate those for the portfolio problem. Another fundamental difference is the complexity of the problem. Finding Ralph is a linear problem: if the GPS is off by a meter, Robert’s search will be off by about a meter. Finding an optimal portfolio requires inversion of the covariance matrix (among other things), which is not linear at all.\(^45\) In the base case data, this complexity translates to an error multiplier of up to 528.

Note the circle centered at point B. It is natural to ask what happens when there is no fence or the GPS estimate is wholly within the yard as in B. In these cases, the Markowitz and RE optimizations do not improve the estimate. However, linear constraints are always present in practice. Moreover, as Jobson and Korkie have shown, the instability of unconstrained MV optimization in the presence of estimation error implies that B is a very poor estimate of true MV optimality.

Our story demonstrates that, properly used, RE optimization is a never worse, more stable, and likely more realistic solution for computing

\(^{45}\) The error associated with this process relates to the condition number of the covariance matrix. The condition number is a measure of how close to singular the covariance matrix is. Inverting a nearly singular covariance matrix is analogous to dividing by a number close to zero.
MV optimized portfolios in practical investment contexts. Because MV optimization is very sensitive to estimation error, the benefits of RE relative to MV optimization are likely to be highly investment significant. It has not escaped our notice that the results do not depend on a quadratic objective. They may apply to any maximization or optimization problem with estimation error in the context of informative constraints.

46. Estimation error always exists in practice. As Chapter 9 will indicate, an index-relative or long-short optimization framework may require a different approach for defining constraints than the traditional asset allocations in this chapter.