ELASTICAE IN KILLING SUBMERSIONS

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Abstract. Classical elastic curves (elastica) are variational objects with many applications in physics and engineering. Elastica in real space forms are well understood, but in other ambient spaces there are few known explicit examples, except geodesics. In this work we study elastica living in the total space of a Killing submersion focusing on those curves whose osculating plane forms a constant angle with the vertical folition (slant elastica). First, we compute the Euler-Lagrange equations for elastica and construct new examples of slant elastica in Killing submersions. Then, we completely classify the two main families of slant elastica in Bianchi-Cartan-Vranceanu ambient spaces (giving also explicit parametrizations).

1. Introduction

In 1691 J. Bernoulli posed the *flexible rod problem* to determine the shape of a flexible rod of given length, subject to external forces at the ends, and in 1694 he obtained the differential equation of the *rectangular* plane elastica (when the external force is orthogonal to the ends of the rod). In his 1744 book on the Calculus of Variations [12], L. Euler classified the plane elastica following an idea of D. Bernoulli, who had suggested to apply the *least action principle* to the mean squared curvature.

The differential equation for plane elastica was solved by L. Euler in Appendix I to [12]. The study of elastica in the other real space forms was initiated in the eighties [14], attracting the interest of many mathematicians. In particular, elastica in simply connected 2-dimensional real space forms were determined in [7], [15], [18], using different approaches. As for the 3-dimensional case, J. Radon analysed the elastica in $\mathbb{R}^3$ at the beginning of the 20th century [23]. Elastica in $\mathbb{S}^3$ were studied in [2], [16], [22] and elastica in hyperbolic 3-space can be treated using the arguments in [17].

More generally, in modern terms, an *elastic curve*, or simply, *elastica* ,is a curve $\alpha : [t_0, t_1] \to M$ immersed in a Riemannian manifold M which is critical for the *bending energy*

$$\int_\alpha \kappa^2 \, ds \,,$$

for variations with "clamped" ends, where κ and s denote the geodesic curvature and arc-length parameter of α, respectively.

Hence, as the universal cover of a real space form M of arbitrary dimension is a real space form $\widetilde{M}$, and the covering projection is a local isometry, a lifting $\tilde{\alpha}$ of an elastica α in M

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is elastic in $\tilde{M}$ and, as noted in [14], [18], we have that $\tilde{\alpha}$ must be contain in a totally geodesic real space form of dimension ≤ 3. So the study of elastica in real space forms M reduces to the case where M is 3-dimensional and simply connected.

On the other hand, little is known about elastica in spaces with nonconstant sectional curvature, and most of these results concern elastic Frenet helices (curves with constant Frenet curvatures). For instance, in [6] any elastic helix in $\mathbb{CP}^2$ is shown to be the image of a translation of a one-parameter subgroup of $\mathbb{SU}(3)$ under the natural projection, and this fact is used to study elastic curves with constant slant in $\mathbb{CP}^2$. As another instance, the main goal of [13] was to construct new families of elastica when the ambient space is a simple Lie group G with a bi-invariant Riemannian metric. For this purpose, a criterion was given for a pointwise product of one-parameter subgroups to be an elastic curve, which was applied to construct large families of new elastic helices in $\mathbb{SU}(3)$.

In this paper we want to study elastica in the total space of a *Killing submersion*. A Riemannian submersion $\pi : \mathbb{E} \to M$ of a 3-dimensional Riemannian manifold $\mathbb{E}$ over a surface M will be called a Killing submersion if its fibers are the trajectories of a complete unit Killing vector field ξ. Killing submersions are determined by two functions on M, its *Gaussian curvature* and the so called *bundle curvature* (for more details, see [11], [20], and section 2). A remarkable family of Killing submersions are the homogeneous 3-spaces. The classification of simply connected 3-dimensional homogeneous spaces is well-known. The dimension of the isometry group must equal 6, 4 or 3. If the isometry group is of dimension 6, M is a complete real space form, i.e. Euclidean space $\mathbb{E}^3$, a sphere $\mathbb{S}^3(c)$, or a hyperbolic space $\mathbb{H}^3(c)$. If the dimension of the isometry group is 4, M is isometric to one of the following: $\mathbb{SU}(2)$, the special unitary group; $\widetilde{\mathbb{SL}}(2,\mathbb{R})$, the universal covering of the real special linear group; Nil_3, the Heisenberg group; or to a Riemannian product $\mathbb{S}^2(c) \times \mathbb{R}$ or $\mathbb{H}^2(c) \times \mathbb{R}$. Finally, if the dimension of the isometry group is 3, M is isometric to a general simply connected Lie group with left-invariant metric. The above family contains the eight "model geometries" appearing in the famous conjecture of Thurston on the classification of 3-manifolds. With the exception of the hyperbolic 3-space, $\mathbb{H}^3(-1)$, simply-connected homogeneous Riemannian 3-manifolds with isometry group of dimension 4 or 6 can be represented by a 2-parameter family $\mathbb{E}(c,b)$, where $c, b \in \mathbb{R}$. These $\mathbb{E}(c,b)$-spaces are 3-manifolds admitting a global unit Killing vector field whose integral curves are the fibers of a certain Riemannian submersion over the simply-connected constant Gaussian curvature surface $M(c)$, and, therefore, determine Killing submersions $\pi : \mathbb{E}(c,b) \to M(c)$ (for more details, see [10]). A local description of these examples can be given by using the so called *Bianchi-Cartan-Vranceanu* spaces (see section 2). As it turns out, the $\mathbb{E}(c,b)$-spaces are the only simply-connected homogeneous 3-manifolds admitting the structure of a Killing submersion [20].

The structure of the paper is as follows. In section 2 we revise some generalities on Killing submersions and compute the Riemannian curvature tensor. In Section 3 *slant curves* are introduced as curves whose osculating plane forms a constant angle with the vertical foliation. It is shown that slant curves family consists of either *Lancret helices* (curves making a constant angle with the vertical foliation) or curves whose torsion equals the bundle curvature. Also, they are characterized here in terms of distinguish curves of *Hopf Cylinders*. Using results of section 2, the Euler-Lagrange equations for curvature dependent energies and, in particular, for elastica are computed in section 4. Then, in section 5, several families of slant elastica, both, with constant and non-constant curvature, in different total spaces of Killing submersions are constructed (specially when the bundle curvature is constant). As it turns out, slant elastica can be divided into two main families: curves with horizontal Frenet unit normal (Lancret helices) and curves with horizontal Frenet binormal. Finally, section 6 gives a complete classification of these classes of elastica in Bianchi-Cartan-Vranceanu spaces along with explicit parametrizations of all of them.

2. Killing submersions

A Riemannian submersion $\pi : \mathbb{E} \to M$ of a 3-dimensional Riemannian manifold $\mathbb{E}$ over a surface M will be called a *Killing submersion* if its fibers are the trajectories of a complete unit Killing vector field ξ (for more details, see [11], [20]). Fibers are geodesics in $\mathbb{E}$ and form a foliation called the *vertical foliation* and denoted by $\mathcal{F}$. Most of the geometry of a Killing submersion is encoded in a pair of functions: G, r. While the first one G represents the Gaussian curvature function of the base surface M, the other one, r, denotes the so called *bundle curvature* which is defined as follows. Since ξ is a (vertical) unit Killing vector field, then it is clear that for any vector field, Z, on $\mathbb{E}$, there exists a function r_Z (which a priori depends on the chosen vector field) such that $\overline{\nabla}_Z\xi = \mathrm{r}_Z Z \wedge \xi$, where $\overline{\nabla}$ denotes the Levi-Civita connection in $\mathbb{E}$. Actually, it is not difficult to see that r_Z does not depend on the vector field Z (see [11] for details) so we get a function $\mathrm{r} \in C^\infty(\mathbb{E})$, the bundle curvature, satisfying

$$\overline{\nabla}_Z\xi = \mathrm{r} Z \wedge \xi \,. \tag{1}$$

The bundle curvature is obviously constant along fibers and consequently it can be seen as a function on the base, $\mathrm{r} \in C^\infty(M)$. In the product spaces $M \times \mathbb{R}$ the projection over the first factor is a Killing submersion, so its bundle curvature is $\mathrm{r} \equiv 0$. More generally, using (1), it is easy to deduce that $\mathrm{r} \equiv 0$ in a Killing submersion if and only if the horizontal distribution in the total space is integrable. From now on, a Killing submersion will be denoted by $\mathbb{E}(\mathrm{G}, \mathrm{r})$.

The existence of a Killing submersion over a simply connected surface M, with a prescribed *bundle curvature*, $\mathrm{r} \in C^\infty(M)$, has been proved in [20]. Uniqueness, up to isomorphisms, is guaranteed under the assumption of simply connectedness for the total space also.

Moreover, the following result provides the existence of Killing submersions with prescribed bundle curvature over arbitrary Riemannian surfaces.

Proposition 1. [4] *Let M be a Riemannian surface and choose any $\mathrm{r} \in C^\infty(M)$. Then there exists a Killing submersion over M with bundle curvature r. In particular it can be chosen to have compact fibers.*

Any Killing submersion is locally isometric to one of the following canonical examples (see [20]) which include, as we will show later, the so called *Bianchi-Cartan-Vranceanu spaces* for suitable choices of the functions λ, a, b.

Example 1 (Canonical examples)**.** *Given an open set $\Omega \subset \mathbb{R}^2$ and $\lambda, a, b \in \mathcal{C}^\infty(\Omega)$ with $\lambda > 0$, the Killing submersion*

$$\pi : (\Omega \times \mathbb{R}, \mathrm{d}s^2_{\lambda,a,b}) \to (\Omega, \mathrm{d}s^2_\lambda), \quad \pi(x, y, z) = (x, y),$$

where

$$\mathrm{d}s^2_{\lambda,a,b} = \lambda^2(\mathrm{d}x^2 + \mathrm{d}y^2) + (\mathrm{d}z - \lambda(a\,\mathrm{d}x + b\,\mathrm{d}y))^2 \tag{2}$$

and

$$\mathrm{d}s^2_\lambda = \lambda^2(\mathrm{d}x^2 + \mathrm{d}y^2)\,,$$

will be called the canonical example associated to (λ, a, b). *The bundle curvature and the Gaussian curvature are given by*

$$\text{(3)} \qquad 2\mathrm{r} = \frac{1}{\lambda^2}\left((\lambda b)_x - (\lambda a)_y\right), \quad \mathrm{G} = -\frac{1}{\lambda^2}\Delta_o(\log\lambda),$$

where Δ_o *represents the Laplacian with respect to the standard metric in the plane.*

Let $\{e_1, e_2\}$ *be the orthonormal frame in* $(\Omega, \mathrm{d}s^2_\lambda)$, *where* $e_1 = \frac{1}{\lambda}\partial_x$ *and* $e_2 = \frac{1}{\lambda}\partial_y$, *and let* $\{E_1, E_2\}$ *be the horizontal lift of* $\{e_1, e_2\}$ *with respect to* π *and* $E_3 = \partial_z$. *Since* π *is the projection over the first two variables, there exist* $a, b \in \mathcal{C}^\infty(\Omega)$ *such that*

$$\text{(4)} \qquad \begin{cases} (E_1)_{(x,y,z)} = \frac{1}{\lambda(x,y)}\partial_x + a(x,y)\partial_z, \\ (E_2)_{(x,y,z)} = \frac{1}{\lambda(x,y)}\partial_y + b(x,y)\partial_z, \\ (E_3)_{(x,y,z)} = \partial_z. \end{cases}$$

Note that $\{E_1, E_2, E_3\}$ *is an orthonormal frame in* $(\Omega \times \mathbb{R}, \mathrm{d}s^2)$ *which can be supposed positively oriented after possibly swapping* e_1 *and* e_2. *Now it is clear that the global frame* (4) *is orthonormal for* $\mathrm{d}s^2$ *if and only if* $\mathrm{d}s^2$ *is the metric given by* (2). *Regardless of the values of the functions* $a, b \in \mathcal{C}^\infty(\Omega)$, *the Riemannian metric given by equation* (2) *satisfies that* π *is a Killing submersion over* $(\Omega, \mathrm{d}s^2_\lambda)$.

Equation (4) *defines a global orthonormal frame* $\{E_1, E_2, E_3\}$ *for* $\mathrm{d}s^2_{\lambda,a,b}$, *where* E_1 *and* E_2 *are horizontal, and* E_3 *is a unit vertical Killing field. It is easy to check that* $[E_1, E_3] = [E_2, E_3] = 0$ *and*

$$[E_1, E_2] = \tfrac{\lambda_y}{\lambda^2}E_1 - \tfrac{\lambda_x}{\lambda^2}E_2 + \left(\tfrac{1}{\lambda^2}(b\lambda_x - a\lambda_y) + \tfrac{1}{\lambda}(b_x - a_y)\right)E_3.$$

Example 2 (Bianchi-Cartan-Vranceanu spaces.)**.** *Particularizing the above construction one can get models for all Killing submersions over* $\mathbb{R}^2$, $\mathbb{H}^2(c)$ *and the punctured sphere* $\mathbb{S}^2_*(c)$. *Given* $c \in \mathbb{R}$, *we define* $\lambda_c \in \mathcal{C}^\infty(\Omega_c)$ *as*

$$\lambda_c(x,y) = \left(1 + \tfrac{c}{4}(x^2+y^2)\right)^{-1},$$

where

$$\Omega_c = \begin{cases} \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < \frac{-4}{c}\}, & \text{if } c < 0, \\ \mathbb{R}^2, & \text{if } c \geq 0. \end{cases}$$

Then, the metric $\lambda_c^2(\mathrm{d}x^2 + \mathrm{d}y^2)$ *in* Ω_c *has constant Gaussian curvature* $\mathrm{G} = c$. *If, in addition, we choose* $a = -\mu y$ *and* $b = \mu x$ *for some real constant* μ, *then one obtains the metrics of the Bianchi-Cartan-Vranceanu spaces* $\mathbb{E}(c, \mu) \equiv \Omega_c \times \mathbb{R}$ ([10, Section 2.3])*:*

$$\lambda_c^2(\mathrm{d}x^2 + \mathrm{d}y^2) + (\mathrm{d}z + \mu\,\lambda_c\,(y\,\mathrm{d}x - x\,\mathrm{d}y))^2.$$

A simple computation, using (3), *gives* $\mu = \mathrm{r}$. *Thus the Bianchi-Cartan-Vranceanu spaces can be seen as the canonical models of Killing submersions with constant bundle curvature and constant Gaussian curvature.*

Now, we want to compute the Riemannian curvature $\overline{R}$ of the total space $\mathbb{E}(\mathrm{G}, \mathrm{r})$ of a Killing submersion $\pi : \mathbb{E}(\mathrm{G}, \mathrm{r}) \to M$ in terms of G and the bundle curvature r. Since the computation is purely local, we will work in a canonical example (see Example 1) associated to some functions $\lambda, a, b \in \mathcal{C}^\infty(\Omega)$ with $\lambda > 0$ and $\Omega \subset \mathbb{R}^2$. Koszul formula yields the Levi-Civita connection in the canonical orthonormal frame $\{E_1, E_2, E_3\}$ given by (4):

$$\text{(5)} \qquad \begin{array}{lll} \overline{\nabla}_{E_1}E_1 = -\frac{\lambda_y}{\lambda^2}E_2, & \overline{\nabla}_{E_1}E_2 = \frac{\lambda_y}{\lambda^2}E_1 + \mathrm{r}E_3, & \overline{\nabla}_{E_1}E_3 = -\mathrm{r}E_2, \\ \overline{\nabla}_{E_2}E_1 = \frac{\lambda_x}{\lambda^2}E_2 - \mathrm{r}E_3, & \overline{\nabla}_{E_2}E_2 = -\frac{\lambda_x}{\lambda^2}E_1, & \overline{\nabla}_{E_2}E_3 = \mathrm{r}E_1, \\ \overline{\nabla}_{E_3}E_1 = -\mathrm{r}E_2, & \overline{\nabla}_{E_3}E_2 = \mathrm{r}E_1, & \overline{\nabla}_{E_3}E_3 = 0. \end{array}$$

Moreover, the frame $\{E_1, E_2\}$ is a basis for the smooth *horizontal distribution* $\mathcal{H}$ of $\mathbb{E}(\mathrm{G}, \mathrm{r})$. The horizontal projection of a vector field U onto $\mathcal{H}$ will be denoted by U^h. A vector U is called *horizontal* if $U = U^h$. A *horizontal curve* is a C^∞ curve whose tangent vector lies in the horizontal distribution. Recall that $\mathcal{H}$ is integrable if and only if $\mathrm{r} = 0$. Denote by

$$J : \mathcal{H} \;\to\; \mathcal{H}\,, \tag{6}$$

the anticlockwise $\frac{\pi}{2}$ rotation. Then from (5) we have

$$\overline{\nabla}_Z \xi = \mathrm{r}(Z \wedge \xi) = -\mathrm{r}J(Z^h)\,, \quad \langle X, J(Y^h)\rangle = -\langle Y, J(X^h)\rangle\,. \tag{7}$$

Another direct computation from (3) and (5) gives

$$\langle \overline{R}(E_j, E_3)E_j, E_3\rangle = -\mathrm{r}^2\,, \qquad \langle \overline{R}(E_1, E_2)E_j, E_3\rangle = -E_j(\mathrm{r})\,, \quad j = 1, 2\,, \tag{8}$$

$$\langle \overline{R}(E_1, E_3)E_2, E_3\rangle = 0\,, \qquad \langle \overline{R}(E_1, E_2)E_1, E_2\rangle = 3\mathrm{r}^2 - \mathrm{G}\,, \tag{9}$$

where G is the Gaussian curvature of M and, in computing the last formula, we have used that $\langle [E_1, E_2], E_3\rangle = 2\mathrm{r}$. Finally, putting $X = X^h + \langle X, E_3\rangle E_3$, $Y = Y^h + \langle Y, E_3\rangle E_3$, $Z = Z^h + \langle Z, E_3\rangle E_3$, $W = W^h + \langle W, E_3\rangle E_3$, using (8), (9) and after a long computation, one has

$$\begin{aligned} \langle \overline{R}(X,Y)Z,W\rangle &= (\mathrm{G} - 3\mathrm{r}^2)\,\{\langle Y, Z\rangle\langle X, W\rangle - \langle X, Z\rangle\langle Y, W\rangle\} \\ &- (\mathrm{G} - 4\mathrm{r}^2)\,\{\langle Y, E_3\rangle\langle Z, E_3\rangle\langle X, W\rangle - \langle X, E_3\rangle\langle Z, E_3\rangle\langle Y, W\rangle \\ &+ \langle X, E_3\rangle\langle Y, Z\rangle\langle E_3, W\rangle - \langle Y, E_3\rangle\langle X, Z\rangle\langle E_3, W\rangle\} \\ &+ \langle Z, J(W^h)\rangle J((X \wedge Y)^h)(\mathrm{r}) + \langle X, J(Y^h)\rangle J((Z \wedge W)^h)(\mathrm{r})\,. \end{aligned} \tag{10}$$

3. Slant curves in Killing submersions

Geodesics of a Killing submersion $\mathbb{E}(\mathrm{G}, \mathrm{r})$ make a constant angle with the vertical foliation spanned by ξ. In fact, if γ is a geodesic, using (1), we have

$$\gamma'\langle \gamma', \xi\rangle = \langle \overline{\nabla}_{\gamma'}\gamma', \xi\rangle + \langle \gamma', \gamma' \wedge \xi\rangle = 0\,,$$

where $\gamma' = \frac{d\gamma}{ds}$. So it seems natural to consider non-geodesic curves which make a constant angle with the vertical foliation (or, equivalently, with ξ). These curves are called *Lancret helices with axes* ξ in the sense of [3]. More generally we shall consider a family of curves, which will be called *slant curves* and are defined as follows.

Take a unit speed C^∞curve immersed in $\mathbb{E}(\mathrm{G}, \mathrm{r})$, $\gamma : I \to \mathbb{E}(\mathrm{G}, \mathrm{r})$. We say that γ is a *Frenet curve* if $\gamma' = \frac{d\gamma}{ds}$ and $\overline{\nabla}_{\gamma'}\gamma'$ are linearly independent at any point. If γ is a Frenet curve, then we have a well defined *Frenet frame* $\{T = \frac{d\gamma}{ds}, N = \frac{\nabla_T T}{|\nabla_T T|}, B = T \times N\}$. As usual κ and τ will denote the associated first and second *Frenet curvatures* (that we simply call *curvature and torsion*). We have the well known *Frenet equations* defining the curvatures

$$\nabla_T T = \kappa N\,, \quad \nabla_T N = -\kappa T + \tau B\,, \quad \nabla_T B = -\tau N\,. \tag{11}$$

Definition 1. *A unit speed non-geodesic curve in* $\mathbb{E}(\mathrm{G}, \mathrm{r})$ *is called a slant curve if its osculating plane makes a constant angle with* ξ*, i.e. the function* $\langle B, \xi\rangle$ *is constant along the curve.*

The following formulas will be useful in what follows. If γ is a Frenet curve, by differentiating $\langle T, \xi\rangle$ along $\gamma(s)$, using (1) and the Frenet equations (11), we have

$$T\langle T, \xi\rangle = \langle \nabla_T T, \xi\rangle = \langle \kappa N, \xi\rangle\,, \tag{12}$$

along $\gamma(s)$. Analogously, by differentiating $\langle N, \xi\rangle$ and $\langle B, \xi\rangle$ we obtain respectively

$$T\langle B, \xi\rangle = (\mathrm{r} - \tau)\langle N, \xi\rangle \,, \tag{13}$$

$$-T\langle N, \xi\rangle = (\mathrm{r} - \tau)\langle B, \xi\rangle + \kappa\langle T, \xi\rangle \,, \tag{14}$$

along $\gamma(s)$. As an immediate consequence of (12)–(14), we have the following lemma

Lemma 1. *Let $\gamma : I \to \mathbb{E}^3(\mathrm{G}, \mathrm{r})$ be a unit speed Frenet curve. Then:*

(1) *the curve γ forms a constant angle with the vertical foliation $\mathcal{F}$ (equivalently is a Lancret helix with axes ξ), if and only if, $N \in \mathcal{H}$. Moreover, in this case it is a slant curve;*

(2) *if γ is a slant curve then either γ is a Lancret helix with axes ξ or $\tau = \mathrm{r}$ along γ.*

This Lemma shows that slant curves generalize the notion of Lancret helices with axes ξ. Obviously, they also include curves with horizontal binormal. A geometric description of both families, Frenet curves with horizontal normal and Frenet curves with horizontal binormal, can be given in terms of *Hopf ξ-cylinders* (i.e., *ξ-equivariant surfaces*) which we recall now.

Let $\pi : \mathbb{E}(\mathrm{G}, \mathrm{r}) \to M$ be a Killing submersion. For any curve, $\beta(u)$, in M (called, *profile curve*), choose a horizontal lift, say $\bar{\beta}(u)$. Since π is Riemannian submersion, both curves have the same speed, so we will assume that both are parameterized by the arc-length. Now, consider the ξ-equivariant surface $S_\beta = \pi^{-1}(\beta)$ in $\mathbb{E}(\mathrm{G}, \mathrm{r})$, which can be parameterized by horizontal lifts of β and ξ-orbits (from now on, simply orbits) via the following map

$$X^\xi(u, v) = \phi_v(\bar{\beta}(u)) \,, \tag{15}$$

where $\{\phi_t : t \in \mathbb{R}\}$ is the one-parameter group generated by ξ. Then S_β is a flat surface which we will call the *Hopf cylinder with profile curve $\beta(s)$*.

Now, it is easy to compute the second fundamental form of a Hopf cylinder (see [11]). Let $\{T_\beta = \beta', N_\beta = JT, \}$ be the Frenet frame of $\beta(u)$ in M. With respect to the parametrization $X^\xi(u, v) = \phi_v(\bar{\beta}(u))$ of the Hopf cylinder, $\bar{T}(u, v) = X^\xi_u(u, v) = d\phi_v(\bar{\beta}'(u))$ is a horizontal lift of T_β along the Hopf cylinder. We denote by $\bar{N}$ the horizontal lift of N_β and choose ξ so that $\{\bar{T}, \bar{N}, \xi\}$ is positively oriented. Thus $\bar{N} = \xi \wedge \bar{T}$ is the unit normal to the Hopf cylinder, and using (1) we have that the second fundamental form of (15) is given by

$$\begin{pmatrix} \kappa_g & -r \\ -r & 0 \end{pmatrix} \tag{16}$$

where κ_g stands for the geodesic curvature of the profile curve in the base M.

In this context, any unit speed curve $\gamma(s)$ in the flat surface $S_\beta = \pi^{-1}(\beta)$, must be of the form $\gamma(s) = X^\xi(\int_o^s \sin\varphi(w)\, dw, \int_o^s \cos\varphi(w)\, dw)$ so that $\gamma' = T = \sin\varphi\, \bar{T} + \cos\varphi\, \xi$. Therefore, the normal curvature of γ in S_β, κ_n is

$$\kappa_n(s) = \langle \overline{\nabla}_T T, \bar{N}\rangle = \langle A_{\bar{N}}(T), T\rangle \,,$$

where $A_{\bar{N}}$ is the shape operator of $S_\beta = \pi^{-1}(\beta)$. Computing the matrix of $A_{\bar{N}}$, with respect to the frame $\{\bar{T}, \xi\}$, we get

$$\kappa_n = \begin{pmatrix} \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \kappa_g & -r \\ -r & 0 \end{pmatrix} \begin{pmatrix} \sin\varphi \\ \cos\varphi \end{pmatrix} = \kappa_g \sin^2\varphi - 2\mathrm{r} \sin\varphi \cos\varphi. \tag{17}$$

Now, we give a description of curves with horizontal binormal. We need another definition for that. We say that a point $\gamma(s_o)$ of a unit speed non-geodesic curve γ in a Killing submersion $\mathbb{E}(\mathrm{G}, \mathrm{r})$ is a *vertical point* if $\gamma'(s_o)$ is vertical. If $\gamma(s)$ has isolated vertical points then for any point which is not vertical there exist a neighbourhood where the restriction $\gamma_1 = \pi\gamma$ is a regular curve in M. The following result must be understood locally (i,e. within any segment of curve between two consecutive vertical points) and generalizes Proposition 4.8 of [5] (which was given for Bianchi-Cartan-Vranceanus spaces)

Proposition 2. *Let γ be a unit speed non-geodesic curve in a Killing submersion $\mathbb{E}(\mathrm{G},\mathrm{r})$ with isolated vertical points. Denote by $\{\gamma' = T, N, B\}$ its Frenet frame and by κ its Frenet curvature. Denote also by $\psi \in [0, \frac{\pi}{2}]$ the angle satisfying $\cos\psi = \langle\gamma', \xi\rangle$ and by κ_g the geodesic curvature of $\gamma_1 = \pi\gamma$ in M (which is the same as the geodesic curvature of γ in the Hopf Cylinder S_{γ_1}). Then, the following assertions are equivalent*

i) *B is horizontal along γ;*
ii) *$\kappa + \psi' = 0$ along γ, where κ is the curvature of γ;*
iii) *γ is an asymptotic line of the Hopf Cylinder S_{γ_1} (i.e. $\kappa_n = 0$);*
iv) *$\kappa_g = -2\mathrm{r}\cot\psi$ along γ.*

Proof. i) $\Leftrightarrow$ ii) easily follows from $T\langle T, \xi\rangle = -\psi'\sin\psi$ and $\langle T, \xi\rangle^2 + \langle N, \xi\rangle^2 + \langle B, \xi\rangle^2 = 1$. Now, denote by $\bar{\gamma}_1$ a horizontal lift of γ_1. Then, $TS_{\gamma_1} = \mathrm{span}\{\xi, \gamma'\} = \mathrm{span}\{\xi, \bar{\gamma}_1\}$. Moreover

$$\overline{\nabla}_T T = \kappa N = \kappa_\gamma N_\gamma + \kappa_n U\,, \tag{18}$$

where κ_γ is the geodesic curvature of γ in S_{γ_1}, N_γ is the unit normal to γ is S_{γ_1} and U is a unit normal to S_{γ_1} along γ. From this equation, we see that B is horizontal $\Leftrightarrow B \perp TS_{\gamma_1} \Leftrightarrow N_\gamma = N \Leftrightarrow \kappa_n = 0$. This shows i) $\Leftrightarrow$ iii). Finally, (17) gives iii) $\Leftrightarrow$ iv). □

At this point, we illustrate some generalities on Lancret helices with axes ξ in a Killing submersion. As before, most of them are extensions of the corresponding properties for Lancret helices in Bianchi-Cartan-Vranceanus spaces proven in [5]. We use previous notation and begin with the following proposition.

Proposition 3. *Let γ be a non-geodesic Lancret helix with axes ξ in $\mathbb{E}(\mathrm{G},\mathrm{r})$. Then there exists an angle $\varphi \in (0, \pi)$ such that*

$$(\mathrm{r} - \tau)\sin\varphi + k\cos\varphi = 0\,. \tag{19}$$

Proof. Since $\langle T, \xi\rangle$ is a non-zero constant there exist an angle $\varphi \in (0, \pi)$ with $\langle T, \xi\rangle = \cos\varphi$. From Lemma 1 $\langle N, \xi\rangle = 0$ and since $\langle T, \xi\rangle^2 + \langle N, \xi\rangle^2 + \langle B, \xi\rangle^2 = 1$ we have $\langle B, \xi\rangle = \sin\varphi$. Finally (14) gives (19). □

We now give a geometric description of Lancret helices with axes ξ in $\mathbb{E}(\mathrm{G},\mathrm{r})$. This description works exactly as that given in [5] for Lancret helices in 3-dimensional homogenous space and, for completeness, we will add the details in the following proposition.

Proposition 4. *The Lancret helices with axis ξ in $\mathbb{E}(\mathrm{G},\mathrm{r})$ are just the geodesics of Hopf cylinders.*

Proof. Let ∇' be the Levi-Civita connection of the induced metric on S_β and let $\gamma(s)$ be a unit speed geodesic in a certain Hopf cylinder, S_β. Then $\nabla'_{\gamma'}\gamma'(s) = 0$ and so we have

$$\frac{d}{ds}\langle\gamma'(s), \xi(\gamma(s))\rangle = \langle\overline{\nabla}_{\gamma'}\gamma'(s), \xi(\gamma(s))\rangle + \mathrm{r}\langle\gamma'(s), \gamma'(s) \times \xi(\gamma(s))\rangle = 0,$$

which shows that the angle that $\gamma(s)$ makes with ξ is constant. In other words, the geodesics of the Hopf cylinders are Lancret helices with axes ξ.

To show the converse, let $\gamma(s)$ be a unit speed Lancret helix with axis ξ, so there exists $\varphi \in \mathbb{R}$, which we may suppose different from zero (otherwise the result is trivial) such that $\langle\gamma', \xi\rangle(s) = \cos\varphi$. Now, define the curve $\beta(s) = \pi(\gamma(s))$ in M and use it as a profile curve to get the Hopf cylinder $S_\beta = \pi^{-1}(\beta)$. It is clear that the original Lancret helix lies in S_β, and, moreover, we have

$$0 = \frac{d}{ds}\langle \gamma', \xi\rangle(s) = \langle \overline{\nabla}_{\gamma'}\gamma', \xi\rangle(s) + \mathrm{r}\langle \gamma', \gamma' \wedge \xi\rangle(s) = \langle \nabla'_{\gamma'}\gamma', \xi\rangle(s),$$

which proves that $\nabla'_{\gamma'}\gamma'(s) = 0$, and so $\gamma(s)$ is a geodesic in $S_\beta = \pi^{-1}(\beta)$. □

Finally, we now want to link the curvature κ of a Lancret helix with axes ξ with the geodesic curvature κ_g of the curve $\beta(s) = \pi(\gamma(s))$ in M. Proceeding as in formulas (16) and (17), let $\{T_\beta = \beta', N_\beta = JT,\}$ be the Frenet frame of $\beta(u)$ in M. With respect to the parametrization $X^\xi(u,t) = \phi_t(\bar\beta(u))$ of the Hopf cylinder, $\bar T(u,t) = X^\xi_u(u,t) = d\phi_t(\bar\beta'(u))$ is a horizontal lift of T_β along the Hopf cylinder. In this context, since $\gamma(s)$ is a geodesic in the flat surface $S_\beta = \pi^{-1}(\beta)$, it must be the image under X^ξ of a certain straight line in the (u,t)-plane. Therefore, there exists a constant φ such that $\gamma(s) = X^\xi((\sin\varphi)\, s, (\cos\varphi)\, s)$. Then we have $T = \sin\varphi\, \bar T + \cos\varphi\, \xi$. We also know that $N = \bar T \wedge \xi$, because it must agree with the unit normal to the Hopf cylinder, and therefore $B = -\cos\varphi\, \bar T + \sin\varphi\, \xi$. Observe that the constant φ is the same as that defined in Proposition 3. Notice also that now the curvature function, $\kappa(s)$, should coincide with the normal curvature which, in turn, was given in (16). Thus we have

$$\kappa = \begin{pmatrix} \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \kappa_g & -r \\ -r & 0 \end{pmatrix} \begin{pmatrix} \sin\varphi \\ \cos\varphi \end{pmatrix} = \kappa_g \sin^2\varphi - 2\mathrm{r}\sin\varphi\cos\varphi. \tag{20}$$

4. Critical curves for curvature energies

We first consider general *curvature energy functionals* acting on spaces of curves satisfying given boundary conditions in $(\mathbb{E}(\mathrm{G},\mathrm{r}), \langle,\rangle)$. We shall denote by $\Omega_{p_op_1}$ the space of smooth immersed curves of $\mathbb{E}(\mathrm{G},\mathrm{r})$, joining two points of it, that is:

$$\Omega_{p_op_1} = \{\beta : [0,1] \to \mathbb{E}(\mathrm{G},\mathrm{r}), \beta(i) = p_i, i \in \{\ 0,1\}, \frac{d\beta}{dt}(t) \neq 0, \forall t \in [0,1]\}, \tag{21}$$

where $p_i \in \mathbb{E}(\mathrm{G},\mathrm{r}), i \in \{0,1\}$, are arbitrary given points of $\mathbb{E}(\mathrm{G},\mathrm{r})$. For a given curve γ in $\Omega_{p_op_1}$, we denote by $T(t)$, $N(t)$ the unit tangent and normal vectors respectively. Also $\kappa^2(t) = \|\nabla_T T\|^2$ will be the squared curvature and κ will be the positive root of κ^2. As usual the arclength parameter is represented by $s \in [0, L]$, L being the length of γ.

We take $P(t)$ a C^∞ function and consider the following *curvature energy functional*

$$\mathbf{\Theta}(\gamma) = \int_\gamma P(\kappa^2) \tag{22}$$

acting on $\Omega_{p_op_1}$. Let us take a curve $\Gamma(w)$ in $\Omega_{p_op_1}$ passing through γ, that is $\Gamma(w,t) = \gamma_w(t) : (-\varepsilon,\varepsilon) \times \mathbb{I} \to \mathbb{E}(\mathrm{G},\mathrm{r})$ is a variation of γ in $\Omega_{p_op_1}$ with $\gamma(0,t) = \gamma(t)$, whose variation vector field along the curve γ is given by $W = W(t) = \frac{\partial\gamma}{\partial w}(0,t)$. The restriction of a *curvature energy functional* to a variation is denoted by the same letter, $\mathbf{\Theta}(w) = \mathbf{\Theta}(\gamma_w(t))$. To compute the first derivative of $\mathbf{\Theta}(w)$ we shall use the following vector fields along γ

$$\begin{aligned} \mathcal{K}(\gamma) &= 2\dot P \cdot \nabla_T T, \\ \mathcal{J}(\gamma) &= \nabla_T \mathcal{K} + \left(4\kappa^2 \dot P - P\right) \cdot T, \\ \mathcal{E}(\gamma) &= \nabla_T \mathcal{J} + 2\dot P \cdot R(\nabla_T T, T)T, \end{aligned} \tag{23}$$

where $\dot P = \frac{dP}{d\kappa^2}$. Then using Lemma 1 and Proposition 2.1 of [18] and integrating by parts one can obtain [1], [2],

Proposition 5. *Let $\Gamma(w,s)=\gamma_w(t)$ be a variation of γ by curves in $\Omega_{p_op_1}$. Under the above conditions and notation, the following formula holds*

$$\frac{d}{dw}\mathbf{\Theta}(w)_{|w=o}=\int_0^L\langle\mathcal{E}(\gamma),W\rangle ds+\mathcal{B}\left[W,\gamma\right]_0^L, \tag{24}$$

where $\mathcal{E}(\gamma)$ is defined in (23) *and the boundary term is given by*

$$\mathcal{B}\left[W,\gamma\right]_0^L=\left[\langle\mathcal{K},\nabla_TW\rangle-\langle\mathcal{J},W\rangle\right]_0^L. \tag{25}$$

Now, we restrict $\mathbf{\Theta}$ to act on spaces of curves which, in addition to the defining conditions of $\Omega_{p_op_1}$, satisfy also a suitable set of boundary conditions. More concretely, we consider $\widehat{\Omega}_{p_op_1}$, the subspace of curves in $\Omega_{p_op_1}$ which also verify: $\frac{d\beta}{dt}(i)=v_i$, where $v_i\in T_{p_i}\mathbb{E}(\mathrm{G},\mathrm{r})$ are fixed tangent vectors to $\mathbb{E}(\mathrm{G},\mathrm{r})$ at $p_i, i\in\{1,2\}$ (although the following computations might be equally applied, for instance, to the case in which $\mathbf{\Theta}$ act on the space of smooth closed curves of $\mathbb{E}(\mathrm{G},\mathrm{r})$). In these cases, the above boundary term vanishes. Thus a critical curve of $\mathbf{\Theta}$ in such spaces will be characterized by the *Euler-Lagrange equation* $\mathcal{E}(\gamma)=0$, in other words, γ is critical, if and only if,

$$2\nabla_T^2\left(\dot{P}\cdot\nabla_TT\right)+\nabla_T\left(\left(4\kappa^2\dot{P}-P(\kappa)\right)\cdot T\right)+2\dot{P}\cdot R(\nabla_TT,T)T=0\,. \tag{26}$$

This means that geodesics in $\widehat{\Omega}_{p_op_1}$ are always critical for $\mathbf{\Theta}$.

From now on, we assume that γ is not geodesic. We are in the right position to state the following theorem.

Theorem 1. *A unit speed Frenet curve $\gamma\in\widehat{\Omega}_{p_op_1}$ is a critical curve for $\mathbf{\Theta}$, if and only if,*

$$P_\kappa''+P_\kappa\left(\kappa^2-\tau^2+\mathrm{r}^2+(\mathrm{G}-4\mathrm{r}^2)\langle B,E_3\rangle^2+2\langle N,JT^h\rangle JB^h(\mathrm{r})\right)-\kappa P=0\,, \tag{27}$$

$$\left(\tau P_\kappa^2\right)'+P_\kappa^2\left((4\mathrm{r}^2-\mathrm{G})\langle N,E_3\rangle\langle B,E_3\rangle+\langle B,JT^h\rangle JB^h(\mathrm{r})-\langle N,JT^h\rangle JN^h(\mathrm{r})\right)=0\,. \tag{28}$$

where $P_\kappa=\frac{dP}{d\kappa}$ and $()'$ denotes derivative with respect to s, the natural parameter of γ.

Proof. From (8) and (9)

$$\langle\overline{R}(N,T)T,N\rangle=\mathrm{r}^2+(\mathrm{G}-4\mathrm{r}^2)\langle B,E_3\rangle^2+2\langle N,JT^h\rangle JB^h(\mathrm{r})\,, \tag{29}$$

$$\langle\overline{R}(N,T)T,B\rangle=(4\mathrm{r}^2-\mathrm{G})\langle N,E_3\rangle\langle B,E_3\rangle+\langle B,JT^h\rangle JB^h(\mathrm{r})-\langle N,JT^h\rangle JN^h(\mathrm{r})\,. \tag{30}$$

Then, by combining these equations with (7), (8), (9), (10), (26), and using the Frenet equations (11) and the linear independence of the Frenet Frame, one can obtain (27) and (28) after some long straightforward computations. □

In the case

$$\mathbf{\Theta}(\gamma)=\int_\gamma\kappa^2$$

is the *bending energy* functional a critical curve is called *elastica*. Curves which are critical for the bending energy for variations with constant length should be critical for the energy $\mathbf{\Theta}(\gamma)=\int_\gamma\kappa^2+\lambda$ (λ being a Lagrange multiplier) and will be called λ-*elastica*. We have the following corollary of Theorem 1.

Corollary 1. *A unit speed Frenet curve $\gamma\in\widehat{\Omega}_{p_op_1}$ is an elastica if and only if*

$$2\kappa''+\kappa^3+2\kappa\left(\mathrm{r}^2-\tau^2+(\mathrm{G}-4\mathrm{r}^2)\langle B,E_3\rangle^2+2\langle N,JT^h\rangle JB^h(\mathrm{r})\right)=0\,, \tag{31}$$

$$(\tau\kappa^2)'+\kappa^2\left((4\mathrm{r}^2-\mathrm{G})\langle N,E_3\rangle\langle B,E_3\rangle+\langle B,JT^h\rangle JB^h(\mathrm{r})-\langle N,JT^h\rangle JN^h(\mathrm{r})\right)=0\,. \tag{32}$$

5. Slant elasticae in Killing submersions

Let now γ be a non-geodesic slant curve in $\mathbb{E}(\mathrm{G},\mathrm{r})$. Assume first that γ is not a Lancret curve, then Lemma 1 implies that $N \in \mathcal{H}$ and $\tau = \mathrm{r}$. The following theorem deals with the case in which B is horizontal. We have

Theorem 2. *Take any surface M, denote by G its Gaussian curvature, and consider any smooth function r on it satisfying that the interior of $\mathrm{r}^{-1}(a)$, O, is not empty for some $a \in \mathbb{R}$. Assume that γ is a unit speed non-geodesic elastic curve in a Killing submersion $\mathbb{E}(\mathrm{G},\mathrm{r})$ with B horizontal and $\pi(\gamma) \subset O$. Then $a = 0$ and γ is also an elastica in a totally geodesic Hopf cylinder.*

Proof. From Lemma 1 if B horizontal, then γ is either a Lancret helix with axes ξ or $\tau = \mathrm{r}$. If it were a Lancret helix with axes ξ, then, again by Lemma 1, N would be horizontal and T vertical. But the latter implies that γ is a vertical geodesic giving a contradiction. Thus we must have $\mathrm{r} = \tau = a$ along γ. Using Corollary 1, the conditions for an elastica read, in this case,

$$2\kappa'' + \kappa^3 = 0 \tag{33}$$

$$2\mathrm{r}\kappa' = 0\,. \tag{34}$$

If $\mathrm{r} \neq 0$ then (34) implies that κ is constant and (33) would imply $\kappa = 0$ which is absurd since we are assuming that γ is not a geodesic. Thus $\mathrm{r} = \tau = a = 0$ and κ is a solution of (33). On the other hand, from Proposition 2 we know that γ is an asymptotic curve in S_{γ_1} where γ_1 is a geodesic of M by Proposition 2. iv). Hence, (16) gives that S_{γ_1} is totally geodesically immersed in $\mathbb{E}(\mathrm{G},\mathrm{r})$ and then the curvature of γ in $\mathbb{E}(\mathrm{G},\mathrm{r})$ and its geodesic curvature as a curve in S_{γ_1} are the same. Since the former satisfies (33) so does the latter. But (33) is precisely the elastica equation in a flat surface [18] and we conclude that γ is also an elastica in the Hopf cylinder S_{γ_1}. □

Corollary 2. *Let γ be a unit speed non-geodesic curve in a Killing submersion $\mathbb{E}(\mathrm{G},\mathrm{r})$ with r constant. Then γ is an elastica with horizontal B, if and only if, $\mathrm{r} = \tau = 0$ and γ is an elastica of a totally geodesic Hopf cylinder.*

Proof. The necessary condition follows from Theorem 2. On the other hand, if $\mathrm{r} = \tau = 0$ and γ is an elastica of a totally geodesic Hopf cylinder S_{γ_1}, then the second fundamental form vanishes, so γ is asymptotic and B is horizontal by Proposition 2. Moreover, the curvature of γ in $\mathbb{E}(\mathrm{G},\mathrm{r})$ and its geodesic curvature in S_{γ_1} coincide. Finally, the elastica equation in a flat surface [18] and $\mathrm{r} = \tau = 0$ imply that γ verify the conditions for an elastica of $\mathbb{E}(\mathrm{G},0)$ in this case, i.e., (33) and (34). □

Example 3. *Assume $\pi : \mathbb{E}(\mathrm{G},0) \to M$ is Killing submersion. For any unit speed geodesic, $\beta(u)$, in M, choose a horizontal lift, say $\bar{\beta}(u)$, and consider the Hopf cylinder $S_\beta = \pi^{-1}(\beta)$ which is totally geodesically immersed in $\mathbb{E}(\mathrm{G},0)$. As we know, it can be parameterized by*

$$X^\xi(u,v) = \phi_v(\bar{\beta}(u))\,,$$

where $\{\phi_t : t \in \mathbb{R}\}$ is the one-parameter group generated by ξ. Take any elastica, α in $\mathbb{R}^2$. Then, $\gamma = X^\xi(\alpha)$ is an elastica in $\mathbb{E}(\mathrm{G},0)$.

Now, we consider non-geodesic elastic Lancret helices with axes ξ in a Killing submersion $\mathbb{E}(\mathrm{G},\mathrm{r})$. To study Lancret helices in a general Killing submersion seems to be a very difficult problem, thus we restrict our attention to the case of Killing submersion with constant bundle curvature r.

Let $\gamma(s)$ be a non-geodesic Lancret helix with axes ξ in a Killing submersion $\mathbb{E}(\mathrm{G},\mathrm{r})$ with r constant. Then there exists an angle $\varphi \in (0,\pi)$ such that $\langle T,\xi\rangle = \cos\varphi$ and $\langle B,\xi\rangle = \sin\varphi$. Using Corollary 1, the conditions for an elastica become in this case

$$2\kappa'' + \kappa^3 + 2\kappa\left(\mathrm{r}^2 - \tau^2 + \sin^2\varphi(G - 4\mathrm{r}^2)\right) = 0\,, \tag{35}$$

$$\left(\tau\kappa^2\right)' = 0\,. \tag{36}$$

From (36) and (19) follows that both κ and τ must be constant, thus we have the following proposition

Proposition 6. *Let $\gamma(s)$ be a non-geodesic Lancret helix with axes ξ in a Killing submersion $\mathbb{E}(\mathrm{G},\mathrm{r})$ with* r *constant and let $\langle T,\xi\rangle = \cos\varphi$, $\varphi \in (0,\pi)$. Then $\gamma(s)$ is an elastica if and only if κ and τ are both constant, the function G is constant along $\gamma(s)$, and they satisfy*

$$\kappa^2 + 2\left(\mathrm{r}^2 - \tau^2 + \sin^2\varphi(G - 4\mathrm{r}^2)\right) = 0\,. \tag{37}$$

We now use the description of Lancret helix as the geodesics of the Hopf cylinder $S_\beta = \pi^{-1}(\beta)$ where $\beta(s) = \pi(\gamma(s))$ is the projection of $\gamma(s)$ in M. Using (19) and (20) in (37) we find that $\gamma(s)$ is an elastica if and only if the curve $\beta(s)$ has constant geodesic curvature satisfying

$$(1 - 3\cos^2\varphi)A^2 - 4\mathrm{r}\cos\varphi A + 2\sin^2\varphi(G - 4\mathrm{r}^2) = 0\,, \tag{38}$$

where

$$A = \kappa_g \sin\varphi - 2\mathrm{r}\cos\varphi\,.$$

A few consequences can be derived from this. We first have

Proposition 7. *Let $\gamma(s)$ be a horizontal non-geodesic curve in a Killing submersion $\mathbb{E}(\mathrm{G},\mathrm{r})$ with* r *constant. Then $\gamma(s)$ is an elastica in $\mathbb{E}(\mathrm{G},\mathrm{r})$, if and only if, the projection $\beta(s) = \pi(\gamma(s))$ is a λ-elastica in M ($\lambda = 8\mathrm{r}^2$) with constant geodesic curvature.*

Proof. Since $\gamma(s)$ is horizontal the angle $\varphi = \pi/2$, then (38) becomes

$$\kappa_g^2 + 2\mathrm{G} - 8\mathrm{r}^2 = 0$$

which is the equation for a λ-elastica in M with $\lambda = 8\mathrm{r}^2$ (see, for example, [18]). □

A natural question is when a geodesic of the Hopf cylinder S_β over a geodesic $\beta(s)$ in M is a non-geodesic elastica in $\mathbb{E}(\mathrm{G},\mathrm{r})$. We have the following answer

Proposition 8. *Let $\beta(s)$ be a geodesic in a surface M such that for a given constant* r *the Gaussian curvature* G *of M is constant along $\beta(s)$ and $-2\mathrm{r}^2 < \mathrm{G}(\beta(s)) < 4\mathrm{r}^2$. Then there exists a Killing submersion $\pi : \mathbb{E}(\mathrm{G},\mathrm{r}) \to M$ such that any geodesic of the Hopf cylinder S_β with slope $m = \sqrt{(4\mathrm{r}^2 - \mathrm{G})/(2\mathrm{r}^2 + \mathrm{G})}$ is a non-geodesic elastica in $\mathbb{E}(\mathrm{G},\mathrm{r})$.*

Proof. The existence of the Killing submersion $\pi : \mathbb{E}(\mathrm{G},\mathrm{r}) \to M$ is guaranteed by Proposition 1. Now, since $\beta(s)$ is a geodesic, from (38) a geodesic of the Hopf cylinder S_β is a non-geodesic elastica in $\mathbb{E}(\mathrm{G},\mathrm{r})$ if and only if

$$6\mathrm{r}^2\cos^4\varphi + (\mathrm{G} - 10\mathrm{r}^2)\cos^2\varphi + (4\mathrm{r}^2 - \mathrm{G}) = 0\,.$$

Solving the latter in $\cos^2\varphi$ and rolling out the solution $\cos^2\varphi = 1$ (which would impliy that $\gamma(s)$ is vertical, hence a geodesic) we find

$$\cos^2\varphi = \frac{2}{3} - \frac{\mathrm{G}}{6\mathrm{r}^2}\,,$$

which is, under the hypothesis $-2\mathrm{r}^2 < \mathrm{G}(\beta(s)) < 4\mathrm{r}^2$, compatible. Then, the corresponding slope is given by

$$m = \frac{\cos\varphi}{\sin\varphi} = \sqrt{\frac{\cos^2\varphi}{1-\cos^2\varphi}} = \sqrt{\frac{4\mathrm{r}^2 - \mathrm{G}}{2\mathrm{r}^2 + \mathrm{G}}}\,.$$

□

Example 4. *Take $\Omega = (\varepsilon_1, \varepsilon_2) \times I$, where I is an open interval. Consider any smooth function $f : (\varepsilon_1, \varepsilon_2) \to \mathbb{R}$ with a critical point $f'(t_o) = 0$, $t_o \in (\varepsilon_1, \varepsilon_2)$ which is not a zero of f. Define on Ω the metric $ds^2 = f^2(x)(dx^2 + dy^2)$ and the curve $\gamma(s) := (t_o, \frac{s}{f(t_o)} + \mu)$, $\mu \in \mathbb{R}$. Then γ is a geodesic in $M = (\Omega, ds^2)$ and the Gaussian curvature of M, G, is constant along γ given by $\delta = -\frac{f_{xx}(t_o)}{f^3(t_o)}$. Finally, choose any constant r satisfying $-2\mathrm{r}^2 < \delta < 4\mathrm{r}^2$ and take any Killing submersion on M, $\mathbb{E}(\mathrm{G}, \mathrm{r})$, with bundle curvature r. Then, Proposition 8 tells us that any geodesic of the Hopf cylinder shaped on γ with slope $m = \sqrt{\frac{4\mathrm{r}^2-\mathrm{G}}{2\mathrm{r}^2+\mathrm{G}}}$ is a non-geodesic elastica in $\mathbb{E}(\mathrm{G}, \mathrm{r})$.*

6. Elasticae in Bianchi-Cartan-Vranceanu spaces

In this section we study non-geodesic unit speed Frenet curves in Bianchi-Cartan-Vranceanu (BCV) spaces which are elastic curves. As we have mentioned in the preliminary section, BCV spaces can be seen as the canonical models of Killing submersions with constant bundle curvature and constant Gaussian curvature. Thus if $c, \mathrm{r} \in \mathbb{R}$ and $\lambda_c = 1/(1 + \frac{c}{4}(x^2 + y^2))$, we define $\mathbb{E}^3(c, \mathrm{r})$ as the following open subset of $\mathbb{R}^3$:

$$\left\{(x, y, z) \in \mathbb{R}^3 \colon \lambda_c > 0\right\},$$

equipped with the metric

$$ds^2 = \lambda_c^2(dx^2 + dy^2) + (dz + \mathrm{r}\lambda_c(y\,dx - x\,dy))^2\,. \tag{39}$$

Cartan in [8] shows that the examples above cover in fact all possible 3-dimensional homogeneous spaces with 4-dimensional isometry group. The BCV family also includes two real space forms, which have 6-dimensional isometry group. The full classification of these spaces is as follows (see, for example, [21]):

- if $c = \mathrm{r} = 0$, then $\widetilde{M}^3(c, \mathrm{r}) \cong \mathbb{E}^3$;
- if $c = 4\mathrm{r}^2 \neq 0$, then $\widetilde{M}^3(c, \mathrm{r}) \cong \mathbb{S}^3\left(\frac{c}{4}\right) \setminus \{\infty\}$;
- if $c > 0$ and $\mathrm{r} = 0$, then $\widetilde{M}^3(c, \mathrm{r}) \cong (\mathbb{S}^2(c) \setminus \{\infty\}) \times \mathbb{R}$;
- if $c < 0$ and $\mathrm{r} = 0$, then $\widetilde{M}^3(c, \mathrm{r}) \cong \mathbb{H}^2(c) \times \mathbb{R}$;
- if $c > 0$ and $\mathrm{r} \neq 0$, then $\widetilde{M}^3(c, \mathrm{r}) \cong \mathrm{SU}(2) \setminus \{\infty\}$;
- if $c < 0$ and $\mathrm{r} \neq 0$, then $\widetilde{M}^3(c, \mathrm{r}) \cong \widetilde{\mathrm{SL}}(2, \mathbb{R})$;
- if $c = 0$ and $\mathrm{r} \neq 0$, then $\widetilde{M}^3(c, \mathrm{r}) \cong \mathrm{Nil}_3$.

The orthonormal frame (4) and the formulae for the Levi Civita connection (5) become, in this case,

$$E_1 = \lambda_c^{-1}\frac{\partial}{\partial x} - \mathrm{r}y\frac{\partial}{\partial z}, \quad E_2 = \lambda_c^{-1}\frac{\partial}{\partial y} + \mathrm{r}x\frac{\partial}{\partial z}, \quad E_3 = \xi = \frac{\partial}{\partial z}. \tag{40}$$

$$\begin{aligned} &\overline{\nabla}_{E_1}E_1=\frac{c}{2}yE_2, && \overline{\nabla}_{E_1}E_2=-\frac{c}{2}yE_1+\mathrm{r}E_3, && \overline{\nabla}_{E_1}E_3=-\mathrm{r}E_2,\\ &\overline{\nabla}_{E_2}E_1=-\frac{c}{2}xE_2-\mathrm{r}E_3, && \overline{\nabla}_{E_2}E_2=\frac{c}{2}xE_1, && \overline{\nabla}_{E_2}E_3=\mathrm{r}E_1,\\ &\overline{\nabla}_{E_3}E_1=-\mathrm{r}E_2, && \overline{\nabla}_{E_3}E_2=\mathrm{r}E_1, && \overline{\nabla}_{E_3}E_3=0. \end{aligned} \tag{41}$$

The conditions of Corollary 1 in this case reduce to

$$2k''+k^3+2k\left(\mathrm{r}^2-\tau^2+(c-4\mathrm{r}^2)\langle B,E_3\rangle^2\right)=0\,, \tag{42}$$

$$(\tau k^2)'+k^2(4\mathrm{r}^2-c)\langle N,E_3\rangle\langle B,E_3\rangle=0\,. \tag{43}$$

We first examine the case when B is horizontal. In this circumstance the following result follows from Corollary 2.

Corollary 3. *Let $\gamma(s)$ be a non-geodesic unit speed Frenet curve in a BCV space $\mathbb{E}^3(c,\mathrm{r})$. Assume that $\gamma(s)$ is an elastica with B horizontal. Then, $\mathrm{r}=0$, that is $\mathbb{E}^3(c,0)=M(c)\times\mathbb{R}$. Moreover, $\gamma(s)$ can be explicitly parametrized in the following way:*

$$\gamma(s)=\left(\beta(u(s)),\int_0^s\cos\psi(v)\,dv\right), \tag{44}$$

where, if we denote by $\mathrm{sn}\,(x,p)$ the Jacobi sine of modulus p,

$$\psi(s)=-2\arcsin\left(\frac{1}{\sqrt{2}}\,\mathrm{sn}\left(\frac{a}{\sqrt{2}}\,s+b,\frac{1}{\sqrt{2}}\right)\right)\,, \tag{45}$$

$a,b\in\mathbb{R}$ and $\beta(u)$ is a unit speed geodesic in $M(c)$ with arc-length function

$$u(s)=\int_0^s\sin\psi(v)\,dv\,. \tag{46}$$

Proof. Corollary 2 tells us that $\mathrm{r}=0$ and that γ must be an elastica in a Hopf cylinder. On the other hand, curves in $\mathbb{E}^3(c,0)=M(c)\times\mathbb{R}$ with horizontal binormal have been totally described in [5], Corollary 4.2. As a result, they can be parametrized by (44), where

$$\psi(s)=-\int_0^s\kappa\,, \tag{47}$$

κ being an arbitrary function playing the role of curvature of γ, and where $\beta(u)$ is a unit speed geodesic in $M(c)$ with arc-length function given by (46). Now, since γ is an elastica, its curvature κ must be a solution of (33). Then $\kappa(s)=a\,\mathrm{cn}\left(\frac{a}{\sqrt{2}}\,s+b,\frac{1}{\sqrt{2}}\right)$, $a,b\in\mathbb{R}$ and $\mathrm{cn}(x,p)$ denoting the Jacobi cosine of modulus p. Then, (45) follows from (47). □

Remark 1. *Observe, that since explicit parametrizations of geodesics in $M(c)$ are well known and $u(s)$, $\int_0^s\cos\psi(v)\,dv$ can be explicitly computed (although they give rise to complicated expression), we have that (44) offers explicit parametrizations of this kind of curves.*

After Corollary 3 and the above Remark, we can assume in the sequel that B is not horizontal.

Proposition 9. *Let $\gamma(s)$ be a non-geodesic unit speed Frenet curve in a BCV space $\mathbb{E}^3(c,\mathrm{r})$. Assume that $\gamma(s)$ is an elastica with $\tau=0$ and B not horizontal.*

- i) *If $4\mathrm{r}^2=c$, then $\mathbb{E}^3(c,\mathrm{r})=\mathbb{S}^3(\frac{c}{4})$ and γ is an elastica in $\mathbb{S}^2(\frac{c}{4})$.*
- ii) *If $4\mathrm{r}^2\neq c$, then $\mathbb{E}^3(c,\mathrm{r})=M(c)\times\mathbb{R}$ and γ is the horizontal lift of an elastica in the 2-real space form $M(c)$.*

Proof. From (43), since $\tau = 0$,

$$(c - 4\mathrm{r}^2)\langle N, E_3\rangle\langle B, E_3\rangle = 0. \tag{48}$$

If $c - 4\mathrm{r}^2 = 0$, then $\mathbb{E}^3(c, \mathrm{r}) = \mathbb{S}^3(\frac{c}{4})$ and, since $\tau = 0$, γ must be included in a totally geodesic $\mathbb{S}^2(\frac{c}{4})$ of $\mathbb{S}^3(\frac{c}{4})$. Moreover, (42) gives that γ satisfies $2\kappa'' + \kappa^3 + (c/2)\kappa = 0$, which means that it is an elastica in $\mathbb{S}^2(\frac{c}{4})$ (see, for example, [18]). If $c - 4\mathrm{r}^2 \neq 0$ and $\langle N, E_3\rangle = 0$, then γ is a Lancret helix with axes E_3 and, from (19), we have

$$0 = \kappa \cot\varphi + \mathrm{r}\,, \tag{49}$$

where $\langle T, E_3\rangle = \cos\varphi$. Since $\kappa > 0$ and we can assume (without loss of generality) that r and $\cot\varphi$ have the same sign, we obtain that the above equation is impossible unless $\mathrm{r} = 0$ and $\varphi = \pi/2$. Then γ is horizontal and satisfies $2\kappa'' + \kappa^3 + 2c\kappa = 0$ from (42). This means that $\gamma = (\beta, t_o)$, $t_o \in \mathbb{R}$, where β is an elastica in $M(c)$. □

We now show that, essentially, the family of elastic slant curve in a BCV space consists of elastic Lancret helix with axes ξ.

Theorem 3. *Let γ be a unit speed non-geodesic elastic slant curve in a BCV space $\mathbb{E}(c, \mathrm{r})$. Assume that B is not horizontal. Then either γ is a Lancret helix with axes ξ or it is a plane elastica in $\mathbb{R}^3$.*

Proof. From Lemma 1 if γ is not a Lancret curve then $\tau = \mathrm{r}$. Integrating (12) and (14), under the hypothesis that $\tau = \mathrm{r}$, we obtain

$$\langle T, \xi\rangle = c_1 \cos \int k + c_2 \sin \int k\,, \quad \langle N, \xi\rangle = -c_1 \sin \int k + c_2 \cos \int k\,. \tag{50}$$

In this case, conditions (42)-(43) for an elastica read

$$2\kappa'' + \kappa^3 + 2\kappa\langle B, \xi\rangle^2(c - 4\mathrm{r}^2) = 0\,, \tag{51}$$

$$2\mathrm{r}\kappa' - \kappa\langle B, \xi\rangle\langle N, \xi\rangle(c - 4\mathrm{r}^2) = 0\,. \tag{52}$$

If $\mathrm{r} = 0$ then (52) implies that $c = 0$. Thus $\mathbb{E}(c, \mathrm{r}) = \mathbb{R}^3$ and the curve γ is a plane curve ($\tau = 0$) with curvature satisfying $2\kappa'' + \kappa^3 = 0$ which is the condition for a plane curve being an elastica.

We show that the assumption $\mathrm{r} \neq 0$ gives a contradiction. By solving (51) in κ' and using (12) and (50) we get

$$\kappa' = \frac{(c - 4\mathrm{r}^2)}{2\mathrm{r}}\langle B, \xi\rangle\langle T, \xi\rangle'\,, \tag{53}$$

which gives, after integration and taking into account (50)

$$\kappa = \frac{(c - 4\mathrm{r}^2)}{2\mathrm{r}}\left(c_1 \cos \int k + c_2 \sin \int k\right)\,. \tag{54}$$

Taking twice the derivative of (54) and using that

$$\langle N, \xi\rangle = \frac{\langle T, \xi\rangle'}{\kappa} = \frac{2\mathrm{r}}{(\mathrm{G} - 4\mathrm{r}^2)\langle B, \xi\rangle}\,,$$

we find

$$\kappa'' = \frac{\kappa'^2}{\kappa} - \kappa^3\,. \tag{55}$$

Combining (51) and (55) we obtain

$$\kappa'^2 = \frac{\kappa^4}{2} - (c - 4\mathrm{r}^2)\langle B, \xi\rangle^2\kappa^2 \,. \tag{56}$$

Next, multiplying (51) by κ' and integrating we deduce

$$\kappa'^2 = -\frac{\kappa^4}{4} - (c - 4\mathrm{r}^2)\langle B, \xi\rangle^2\kappa^2 + C\,, \tag{57}$$

where C is a real constant. Finally (56) and (57) imply that κ is constant and thus (52) becomes

$$\kappa(c - 4\mathrm{r}^2)\langle B, \xi\rangle\langle N, \xi\rangle = 0\,.$$

Since the curve γ is a non-geodesic slant curve which is not a Lancret helix and B is not horizontal, the latter equation can be satisfied only when $c = 4\mathrm{r}^2$. But then (56) would become $\kappa^3 = 0$ which is absurd. □

For what follows we will call a *Frenet helix* a Lancret curve with axis ξ with constant curvature and constant torsion. Frenet helices in BCV spaces (with $c \neq 0$) can be nicely parametrized as shown in the following lemma which is an adapted version of Theorem 5.6 in [9], thus we omit the proof.

Lemma 2. *Let $\gamma(s) = (x(s), y(s), z(s))$ be a non-geodesic non-horizontal unit speed Frenet helix in a BCV space $\mathbb{E}^3(c, \mathrm{r})$, with $c \neq 0$, and denote by κ its curvature. Then, the parametric equations of γ are of the following three types.*

(a) *If $\langle T, E_1\rangle$ is not constant*

$$\begin{cases} x(s) = \mu\sin\varphi\sin\beta(s) + \mu_1\,, \quad \mu, \mu_1 \in \mathbb{R},\ \mu > 0, \\ y(s) = -\mu\sin\varphi\cos\beta(s) + \mu_2\,, \quad \mu_2 \in \mathbb{R}, \\ z(s) = \dfrac{2\mathrm{r}}{c}\beta(s) + \dfrac{1}{c}\big[(c - 4\mathrm{r}^2)\cos\varphi - 2\mathrm{r}\dfrac{\kappa}{\sin\varphi}\big]s, \end{cases}$$

where $\cos\varphi = \langle T, E_3\rangle$, $\beta(s)$ is a non-constant solution of the ODE

$$\beta' + \frac{c}{2}\sin\varphi(\mu_2\cos\beta - \mu_1\sin\beta) = \frac{c}{2}\mu\sin^2\varphi + 2\mathrm{r}\cos\varphi + \frac{k}{\sin\varphi}\,,$$

and the constants satisfy

$$\mu_1^2 + \mu_2^2 = \frac{4\mu}{c}\left\{\left(2\mathrm{r}\cos\varphi + \frac{\kappa}{\sin\varphi} - \frac{1}{\mu}\right) + \frac{c}{4}\mu\sin^2\varphi\right\}.$$

(b) *If $\langle T, E_1\rangle = constant \neq 0$ and $\langle T, E_2\rangle = constant \neq 0$*

$$\begin{cases} x(s) = x(s), \\ y(s) = x(s)\tan\beta + \mu, \\ z(s) = \dfrac{1}{c}\left[(c - 4\mathrm{r}^2)\cos\varphi - 2\mathrm{r}\,\dfrac{\kappa}{\sin\varphi}\right]s + \mu_1\,, \quad \mu_1 \in \mathbb{R}, \end{cases}$$

where $\cos\varphi = \langle T, E_3\rangle$, β is a constant such that $\langle T, E_1\rangle = \sin\varphi\cos\beta$,

$$\mu = \frac{2\kappa + 4\mathrm{r}\sin\varphi\cos\varphi}{c\sin^2\varphi\cos\beta}\,,$$

and $x(s)$ is a solution of the following ODE:

$$x' = \big(1 + \frac{c}{4}[x^2 + (x\tan\beta + \mu)^2]\big)\sin\varphi\cos\beta\,.$$

(c) *If* $\langle T,E_1\rangle\langle T,E_2\rangle=0$*, up to interchange of* x *with* y*,*

$$\begin{cases} x(s)=x_0=\mp\dfrac{2\kappa+4\mathrm{r}\sin\varphi\cos\varphi}{c\sin^2\varphi} \\ y(s)=y(s) \\ z(t)=\dfrac{1}{c}\left[(c-4\mathrm{r}^2)\cos\varphi-2\mathrm{r}\,\dfrac{\kappa}{\sin\varphi}\right]t+\mu, \quad \mu\in\mathbb{R} \end{cases}$$

where $y(s)$ *is a solution of the following ODE:*

$$y'=\pm\big(1+\frac{c}{4}[x_0^2+y^2]\big)\sin\varphi\,.$$

From Lemma 2 the parametrization of a Frenet helix in a BCV space (with $c\neq 0$) is determined by two constants: the value of its curvature κ and the angle $\varphi=\langle T,\xi\rangle$. In the following theorem we will show that an elastic Lancret helix with axes ξ is a Frenet helix and we will describe the two constants κ and φ.

Theorem 4. *Let* γ *be a non-geodesic unit speed Frenet curve in a BCV space* $\mathbb{E}^3(c,\mathrm{r})$ *with* $\tau\neq 0$*. Assume that* $\gamma(s)$ *is a free elastic Lancret helix with axes* ξ*. Then,* $\gamma(s)$ *s a Frenet helix and* $\varphi=\langle T,\xi\rangle$ *must satisfy one of the following conditions*

(1) *Case* $\mathrm{r}=0$*. In this case we have*
   i) *If* $c=0$*, then* $\sin^2\varphi=2/3$*.*
   ii) *If* $c>0$*, then* $\sin^2\varphi<2/3$*.*
   iii) *If* $c<0$*, then* $\sin^2\varphi>2/3$*.*

(2) *Case* $\mathrm{r}\neq 0$*. In this case we have*
   i) *If* $c\leq 4\mathrm{r}^2$*, then* $\sin^2\varphi\geq 2/3$*.*
   ii) *If* $c>4\mathrm{r}^2$*, then* $\sin^2\varphi<\frac{\lambda-5+\sqrt{1-4\lambda+\lambda^2}}{3(\lambda-4)}<1$*, where* $\lambda=\frac{c}{4\mathrm{r}^2}$*.*

Moreover, in all above cases, γ *is a Frenet helix whose curvature is given as a positive root of the polynomial* $f(x):=(1-\cot^2\varphi)\,x^2-4\mathrm{r}\cot\varphi\,x+2(c-4\mathrm{r}^2)\sin^2\varphi$ *and whose torsion is determined by* $\tau=\kappa\cot\varphi+\mathrm{r}$*. Conversely, choose real numbers* $c,\mathrm{r}\in\mathbb{R},\mathrm{r}\geq 0$*, and an angle* $0<\varphi\leq\frac{\pi}{2}$ *satisfying one of the above conditions. Let* γ *be Frenet helix making a constant angle* φ *with* ξ *such that its curvature* κ *is a positive root of* $f(x)$ *and whose torsion is given by* $\tau=\kappa\cot\varphi+\mathrm{r}$*. Then,* γ *is a free elastica in* $\mathbb{E}^3(c,\mathrm{r})$*.*

Proof. From (19), since γ is a Lancret helix with axes ξ, then

$$\tau=\kappa\cot\varphi+\mathrm{r}\,, \tag{58}$$

and therefore, using (42) and (43), we have that γ is an elastica in $\mathbb{E}^3(c,\mathrm{r})$, if and only if, its curvature and torsion, κ and τ, are constants (Frenet helix) satisfying

$$\kappa^2+2\mathrm{r}^2-2\tau^2+2(c-4^2)\sin^2\varphi=0\,, \tag{59}$$

$$\tau=\kappa\cot\varphi+\mathrm{r}\,. \tag{60}$$

This system is equivalent to

$$\big(1-\cot^2\varphi\big)\,\kappa^2-4\mathrm{r}\cot\varphi\kappa+2(c-4\mathrm{r}^2)\sin^2\varphi=0\,, \tag{61}$$

$$\tau=\kappa\cot\varphi+\mathrm{r}\,, \tag{62}$$

and κ must be positive. Applying Descartes' rule of signs to the polynomial $f(x)$ we easily get claims in cases (1) i)–iii) and case (2) i). If we use again Descartes' rule in case (2) ii), we have that $f(x)$ has either 0 or 2 positive real roots. Hence if the discriminant Δ of $f(x)$ is non-negative $f(x)$ is going to have 2 positive real roots. Multiplying Δ by $\sin^2\varphi$, we obtain $q(x)=4\mathrm{r}^2+4(c-5\mathrm{r}^2)x-6(c-4\mathrm{r}^2)x^2$, where we are using $x=\sin^2\varphi$. The roots of $q(x)$ are $\alpha_\pm = \frac{c-5\mathrm{r}^2\pm\sqrt{\mathrm{r}^4-4\mathrm{r}^2c^2}}{3(c-4\mathrm{r}^2)}$. Since, $c > 4\mathrm{r}^2$, one can check that $\alpha_- < 0 < \alpha_+ < 1$. Hence, $q(x)$ is a parabola joining $(0, 4\mathrm{r}^2)$ and $(\alpha_+, 0)$. Now, 2 ii) follows. The converse is clear. □

Although Theorem 4 remains valid when $c = 0$, in Lemma 2 we cannot put $c = 0$ which corresponds to the case of the Heisenberg group Nil_3, so we will consider this case separately .

We recall that the Heisenberg group is the Lie group $(\mathbb{R}^3; *)$, where the product $*$ is defined by

$$(z, x_3) * (w, y_3) := (z + w, x_3 + y_3 - \mathrm{r}\,\mathrm{Im}(z\overline{w})) . \tag{63}$$

Here we are using the notation $x = (x_1, x_2, x_3) = (z, x_3) \in \mathrm{Nil}_3$, with $z = x_1 + \imath x_2 \in \mathbb{C}$ and $x_3 \in \mathbb{R}$. In this case the vector fields (40) become

$$E_1 = \frac{\partial}{\partial x_1} - \mathrm{r}x_2\frac{\partial}{\partial x_3}, \quad E_2 = \frac{\partial}{\partial x_2} + \mathrm{r}x_1\frac{\partial}{\partial x_3}, \quad E_3 = \frac{\partial}{\partial x_3}\,, \tag{64}$$

and form a basis of left invariant vector fields. Moreover, the metric (39) is left invariant.

Now we derive the explicit parametrization of a slant elastica in Nil_3. We point out that, in order to obtain a good description of the elastica in Nil_3, it is enough to find parameterizations for those elastica starting at $(0, 0, 0)$ and then use left translations by the multiplication law (63). Then we have,

Theorem 5. *Let $\gamma(s)$ be a non-geodesic unit speed elastic curve in the Heisenberg group starting at $(0, 0, 0)$. Assume that γ is a slant curve. Then, $\gamma(s) = (\gamma_1(s), \gamma_2(s), \gamma_3(s))$ can be parameterized in the following way*

$$\begin{aligned}
\gamma_1(s) &= \frac{\sin\varphi}{A}\sin(As + b) + \mu_1\,, \\
\gamma_2(s) &= \frac{-\sin\varphi}{A}\cos(As + b) + \mu_2\,, \\
\gamma_3(s) &= (\cos\varphi + \frac{\sin^2\varphi}{2A})s - \frac{\mu_1\sin\varphi}{2A}\cos(As + b) - \frac{\mu_2\sin\varphi}{2A}\sin(As + b) + \mu_3\,,
\end{aligned} \tag{65}$$

where, $\varphi = \langle T, \xi\rangle$, $\sin^2\varphi > \frac{2}{3}$, $b \in \mathbb{R}$, and

$$A = 2\mathrm{r}\left(\cos\varphi + \frac{\left(5 + 3\cos 4\varphi + 4\cos\varphi\,(3 - 2\cos 2\varphi + 3\cos 4\varphi)^{\frac{1}{2}}\right)^{\frac{1}{2}}}{1 + 3\cos 2\varphi}\right), \tag{66}$$

$$\mu_1 = -\frac{\sin b\sin\varphi}{A}; \quad \mu_2 = \frac{\cos b\sin\varphi}{A}; \quad \mu_3 = 0\,. \tag{67}$$

Conversely, choose any $\varphi \in [0, \pi)$ satisfying $\sin^2\varphi > \frac{2}{3}$, $b \in \mathbb{R}$ and consider the curve γ in the Heisenberg group parameterized by (65) and satisfying (66), (67). Then γ is an elastic Frenet helix with slope φ.

Proof. We omit the lengthly but tedious straightforward computations. By differentiating $\langle T, E_j\rangle$, $j = 1, 2$, and using the identities (41), we have

$$\begin{aligned}
T\langle T, E_j\rangle &= \kappa\langle N, E_j\rangle + \sum_{i=1}^{3}\langle T, E_i\rangle\langle T, \nabla_{E_i}E_j\rangle \\
&= \kappa\langle N, E_j\rangle + (-1)^j\langle T, E_3\rangle\langle T, E_{j+(-1)^{j+1}}\rangle\,,
\end{aligned} \tag{68}$$

for $j = 1, 2$. From Lemma 1 and (12) we have that $\langle T, E_3 \rangle = \cos\varphi$ is constant along γ and also $\sin^2\varphi > \frac{2}{3}$ by Theorem 4. Then, $\langle B, E_3 \rangle = \sin\varphi$ is also constant along γ, which is not zero because otherwise (14) would imply that γ is a geodesic. From Lemma 1 and (36) we have that the curvature and torsion of γ are constants satisfying (19)

$$\tau = \kappa \cot\varphi + \mathrm{r}. \tag{69}$$

Moreover, by using (35) we have

$$\kappa^2 = 2(\tau^2 - \mathrm{r}^2) + 8\mathrm{r}^2 \sin^2\varphi \,. \tag{70}$$

Now, since the Frenet frame $\{T, N, B\}$ is orthonormal, there exist a smooth function $\delta(s)$ such that

$$\langle T, E_1 \rangle = \sin\varphi \cos\delta(s) \,, \qquad \langle T, E_2 \rangle = \sin\varphi \sin\delta(s) \,, \qquad \langle T, E_3 \rangle = \cos\varphi \,. \tag{71}$$

Analogously, since $B = T \wedge N$, and after some lengthly but elementary computations, one can see that,$\langle N, E_1 \rangle = \cos(\delta(s) + \frac{\pi}{2})$, $\langle N, E_2 \rangle = \sin(\delta(s) + \frac{\pi}{2})$, $\langle N, E_3 \rangle = 0$, $\langle B, E_1 \rangle = \cos\varphi \cos(\delta(s) + \pi)$, $\langle B, E_2 \rangle = \cos\varphi \sin(\delta(s) + \pi)$, $\langle B, E_3 \rangle = \sin\varphi$. Therefore, by combining these equalities with (68), for $j = 1, 2$, and (71), one can see that

$$\delta(s) = As + b \,, \tag{72}$$

where $A = \frac{2\kappa + \sin 2\varphi}{2\sin\varphi}$. By using, (69) and (70), we see that A must verify (66). Finally, from (64) and (71) we get the result. ☐

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